## Round 1, UNM-PNM Statewide High School Mathematics Contest 2025-2026, Solutions

Dear Students,

If you have suggestions for the Contest, or if you have different solutions to any of this year's first-round problems, please contact us either by ordinary mail or email.


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Please remember that you can find information on past contests at http://mathcontest.unm.edu. We express our gratitude to Coach Sean Choi for running online strategy and solution sessions for the first-round. We also thank Sean and Bill Cordwell for sharing their solutions with us. Finally, thanks to all participants, their teachers, and families. You are an inspiration to us!

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**1**. How many digits are there in the integer $2^{2029} \times 5^{2022}$?

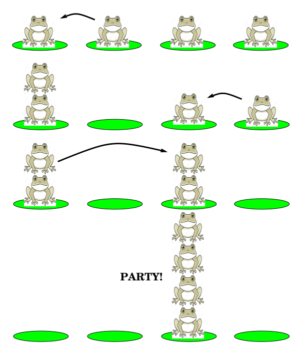
**ANSWER:** $\boxed{2025}$

**Solution.** We notice that

$$2^{2029} \times 5^{2022} = 2^7 \times 2^{2022} \times 5^{2022} = 2^7 \times (2 \times 5)^{2022} = 128 \times 10^{2022}.$$

$10^{2022}$ has 2023 digits, since $10^1 = 10$ has two, $10^2 = 100$ has three, etc. Now, $128 \times 10 = 1280$ has four, $128 \times 10^2 = 12800$ has five, etc. Therefore, $128 \times 10^{2022}$ has 2025 digits.

**2**. A single frog sits on each of four lily pads in a row. One of the frogs then jumps onto the back of a neighbor, either to the left or right. This leaves a configuration with one empty pad, and one pad supporting a group of two frogs. The jumping continues subject to the rules that one frog must jump to a neighboring pad, two frogs must jump two pads as a group, and three frogs must jump three pads as a group. Frogs will not jump onto an empty pad, nor will a group split up once formed. The frogs all seek to occupy a single lily pad, a frog party! The figure shows one sequence of valid jumps resulting in a party. How many different jumping sequences result in a party?




**ANSWER:** $\boxed{14}$

**Solution.** This problem was inspired by the Math Teacher Circle led by Blythe Tipton (Eisenhower Middle School) at UNM on 4 September 2025. We use brute force, exhausting all possible cases. To describe the sequence of jumps, we use the labels (frog;pad), and, as more frogs pile up on one pad, we write, for example, ([frogA,frogB];padB) to mean the frog on pad A has joined the frog on pad B. We start with $(1;1)$, $(2;2)$, $(3;3)$, $(4;4)$, and the sequence shown in the figure is

$$(1;1)(2;2)(3;3)(4;4) \to ([2,1];1)(3;3)(4;4) \to ([2,1];1)([4,3];3) \to ([2,1,4,3];3).$$

We begin by considering all possible first jumps.

$$(1;1)(2;2)(3;3)(4;4) \to ([1,2];2)(3;3)(4;4)$$
$$(1;1)(2;2)(3;3)(4;4) \to ([2,1];1)(3;3)(4;4)$$
$$(1;1)(2;2)(3;3)(4;4) \to (1;1)([2,3];3)(4;4)$$
$$(1;1)(2;2)(3;3)(4;4) \to (1;1)([3,2];2)(4;4)$$
$$(1;1)(2;2)(3;3)(4;4) \to (1;1)(2;2)([3,4];4)$$
$$(1;1)(2;2)(3;3)(4;4) \to (1;1)(2;2)([4,3];3)$$

By symmetry, we now need only explore subsequent jumps starting from the first three of the above initial jumps, *but must then remember to adjust our results below with an appropriate factor of 2.* Indeed, if in the last three initial jumps we send $1 \to 4$, $2 \to 3$, $3 \to 2$, $4 \to 1$, and then reorder the pads, we recover the first three initial jumps. For example, with this flipping reassignment, $(1;1)(2;2)([4,3];3)$ becomes $(4;4)(3;3)([1,2];2)$, and so $([1,2];2)(3;3)(4;4)$ upon reordering. This is the first listed initial jump. Exploitation of the symmetry is key in this problem, or else the casework doubles.

Therefore, consider restarting from one of the following three configurations.

**Case 1.** $\boxed{([1,2];2)(3;3)(4;4)}$ The possible jumps are

$$\begin{aligned}
([1,2];2)(3;3)(4;4) &\to (3,3)([1,2,4];4)\\
([1,2];2)(3;3)(4;4) &\to ([3,1,2];2)(4;4)\\
([1,2];2)(3;3)(4;4) &\to ([1,2];2)([3,4];4)\\
([1,2];2)(3;3)(4;4) &\to ([1,2];2)([4,3];3).
\end{aligned}$$

The first possibility on the right can only yield a party on pad 4. The second possibility can not yield a party, since going from pad 2 to 4 or vice versa requires 2 jumps, but the number of frogs is 3 and 1 on these pads. The third can yield parties on pads 2 or 4. The fourth can not yield a party, since we have 2 groups of 2 frogs on adjacent pads. Summing up, for this case parties on pads 2 and 4 are possible, on pad 2 by 1 sequence and on pad 4 by 2 sequences. **3 possible sequences**

**Case 2.** $\boxed{([2,1];1)(3;3)(4;4)}$ The possible jumps are

$$\begin{aligned}
([2,1];1)(3;3)(4;4) &\to ([2,1,3];3)(4;4)\\
([2,1];1)(3;3)(4;4) &\to ([2,1];1)([3,4];4)\\
([2,1];1)(3;3)(4;4) &\to ([2,1];1)([4,3];3).
\end{aligned}$$

The first possibility on the right can yield a party only on pad 3. The second possibility cannot yield a party. The third possibility can yield parties on pads 1 and 3. Summing up, for this scenario the possibilities are parties on pad 1 by 1 sequence and on pad 3 by 2 sequences. **3 possible sequences**

**Case 3.** $\boxed{(1;1)([2,3];3)(4;4)}$ The possible jumps are

$$\begin{aligned}
(1;1)([2,3];3)(4;4) &\to ([2,3,1];1)(4;4)\\
(1;1)([2,3];3)(4;4) &\to (1;1)([4,2,3];3).
\end{aligned}$$

The first possibility on the right can only yield a party on pad 4, and the second cannot yield a party. Summing up, only a party on pad 4 is possible by 1 sequence. **1 possible sequence**

Collecting the results for these three cases, we determine that a party on pad 1 can be achieved by 1 sequence, on pad 2 by 1 sequence, on pad 3 by 2 sequences, and on pad 4 by 3 sequences. In all, 7 possible sequences lead to parties. Multiplying by the aforementioned factor of 2, we find 14 possible sequences.

**3**. In the following equation $a$, $b$, $c$, and $d$ are positive integers. Find $a+b+c+d$.

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d}}} = \frac{2025}{671}$$

**ANSWER:** $\boxed{70}$

**Solution 1.** We repeatedly use the division algorithm, starting with the numerator of the given fraction. Precisely, $2025 = 3 \times 671 + 12$, dividing by 671; $671 = 12 \times 55 + 11$, dividing by 12; $12 = 1 \times 11 + 1$, dividing by 11. Written out, these steps are

$$\frac{2025}{671} = 3 + \frac{12}{671} = 3 + \cfrac{1}{\cfrac{671}{12}} = 3 + \cfrac{1}{55 + \cfrac{11}{12}} = 3 + \cfrac{1}{55 + \cfrac{1}{\cfrac{12}{11}}} = 3 + \cfrac{1}{55 + \cfrac{1}{1 + \cfrac{1}{11}}}.$$

Therefore, $a + b + c + d = 3 + 55 + 1 + 11 = 70$.

**Solution 2.** This solution is the same as the previous one, except with an inequality given at each step. Since, according to the problem statement, $c, d \geq 1$, we have $1 < c + 1/d < \infty$, so that $0 < 1/(c + 1/d) < 1$. Then, since $b \geq 1$, we have $1 < b + 1/(c + 1/d) < \infty$, so that $0 < 1/(b + 1/(c + 1/d)) < 1$. Therefore, since $2025/671 = 3 + 12/671$, we must have $a = 3$ and $1/(b + 1/(c + 1/d)) = 12/671$. Next,

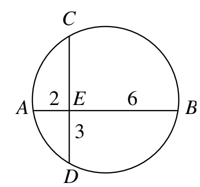
$$b + \frac{1}{c + 1/d} = \frac{671}{12} = 55 + \frac{11}{12}.$$

Whence $b = 55$ and $1/(c + 1/d) = 11/12$. Finally then,

$$c + \frac{1}{d} = \frac{12}{11} = 1 + \frac{1}{11}.$$

From here, we infer $c = 1$ and $d = 11$. So $a + b + c + d = 3 + 55 + 1 + 11 = 70$.

4. Chords $AB$ and $CD$ across the shown circle intersect at $E$ and are perpendicular to each other. If the segments $AE$, $EB$, and $ED$ have lengths 2, 6, and 3, respectively, then what is the square of the circle's diameter?




**ANSWER:** 65

**Solution 1.** From the diagram the product $|AE| \cdot |EB| = 2 \cdot 6 = 12$. Therefore, by the *Intersecting Chords Theorem*, the product $|DE| \cdot |EC| = 3|EC|$ is also equal to 12, so that $|EC| = 4$. The horizontal diameter meets perpendicularly and bisects the segment $DC$. Whence the horizontal diameter lies $\frac{1}{2}$ unit above the chord $AB$. Likewise, the vertical diameter lies 2 units to the right of the chord $DC$. So we have a right triangle whose hypotenuse is the radius $r$, and whose legs have lengths $\frac{1}{2}$ and 4; see Fig. 1 left. Therefore,

$$r = \sqrt{16 + \frac{1}{4}} = \frac{\sqrt{65}}{2}.$$

The diameter has length $\sqrt{65}$.


FIGURE 1

**Solution 2.** Suppose that you did not know the neat geometric fact used in the first solution. Let $O$ denote the center of the circle, with $r$ its radius. Respectively, let $F$ and $H$ denote the feet of the segments drawn from the center $O$ which perpendicularly meet the segments $AB$ and $CD$; see Fig. 1 right. Note that $F$ and $H$ bisect $AB$ and $CD$; indeed, for example, the right triangles $\triangle OHC$ and $\triangle OHD$ share a common leg and have the same hypotenuse $r$. Since $|AB| = 8$, we then have $|FB| = |AF| = 4$, and, with $|AE| = 2$, that $|EF| = 2$. Now $\square FEHO$ is a rectangle, so that $|HO| = |EF| = 2$ and $|FO| = |EH|$. Furthermore, $|OA| = |OD| = r$. Introducing the variable $h = |FO|$, the length $|DH| = |DE| + |EH| = 3 + h$. Consider now the right triangles $\triangle AFO$ and $\triangle DHO$, with the *Pythagorean Theorem* applied to each, respectively yielding the equations

$$r^2 = h^2 + 4^2, \qquad r^2 = (3 + h)^2 + 2^2.$$

Whence $h^2 + 16 = h^2 + 6h + 13$, and, upon canceling $h^2$ on both sides, $h = \frac{1}{2}$. Now we solve for the radius

$$r = \sqrt{\frac{1}{4} + 4^2} = \sqrt{\frac{1 + 64}{4}} = \sqrt{\frac{65}{4}} = \frac{\sqrt{65}}{2}.$$

Therefore the diameter of the circle is $2r = \sqrt{65}$.

**Remark:** Similar reasoning establishes the *Intersecting Chords Theorem* used in the first solution: Consider two chords, $AB$ and $CD$, of a circle which intersect perpendicularly at the point $E$. Then $|AE| \cdot |EB| = |CE| \cdot |ED|$.

**5**. Based on the following *logic table*, with entries T (True) and F (False), what is the probability that the nested statement $\big((p \to q) \to r\big)$ is T? Here each of $p$, $q$, and $r$ are either T or F. Express this probability as a whole fraction in lowest terms, and as your answer report the sum of this fraction's numerator and denominator.

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

**ANSWER:** 13

**Solution 1.** Since $p$, $q$, and $r$ can only be T and F, there are $2^3 = 8$ possible nested statements $(p \to q) \to r$. Namely,

$$\begin{array}{ll} (\mathrm{T} \to \mathrm{T}) \to \mathrm{T} & (\mathrm{F} \to \mathrm{T}) \to \mathrm{T} \\ (\mathrm{T} \to \mathrm{F}) \to \mathrm{T} & (\mathrm{F} \to \mathrm{F}) \to \mathrm{T} \\ (\mathrm{T} \to \mathrm{T}) \to \mathrm{F} & (\mathrm{F} \to \mathrm{T}) \to \mathrm{F} \\ (\mathrm{T} \to \mathrm{F}) \to \mathrm{F} & (\mathrm{F} \to \mathrm{F}) \to \mathrm{F}. \end{array}$$

Using the table to evaluate the expression within each parentheses in these expressions, we get

$$\begin{array}{ll} \mathrm{T} \to \mathrm{T} & \mathrm{T} \to \mathrm{T} \\ \mathrm{F} \to \mathrm{T} & \mathrm{T} \to T \\ \mathrm{T} \to \mathrm{F} & \mathrm{T} \to \mathrm{F} \\ \mathrm{F} \to \mathrm{F} & \mathrm{T} \to \mathrm{F}. \end{array}$$

Finally, use of the table to evaluate each of the preceding expressions yields

$$\begin{array}{ll} \mathrm{T} & \mathrm{T} \\ \mathrm{T} & \mathrm{T} \\ \mathrm{F} & \mathrm{F} \\ \mathrm{T} & \mathrm{F}. \end{array}$$

Therefore, $\frac{5}{8}$ is the probability that the expression $(p \to q) \to r$ is T.

**Solution 2.** The same solution can be presented in tabular form. The logic table for the implication $p \to q$ can be paraphrased as stating that the implication is False (F) only when the hypothesis ($p$) is True (T) and the conclusion ($q$) is False (F). With this principle, we write down the logic table corresponding to $(p \to q) \to r$. We have 3 variables $(p, q, r)$ and 8 possible logic values. In the 8 columns of the table corresponding to the variables $(p, q, r)$, we have listed all possible combinations of the logic values.

| $p$ | $q$ | $p \to q$ | $r$ | $(p \to q) \to r$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | T |
| F | T | T | T | T |
| F | F | T | T | T |
| T | T | T | F | F |
| T | F | F | F | T |
| F | T | T | F | F |
| F | F | T | F | F |

5 scenarios out of 8 possible scenarios yield True (T); therefore, the probability that $(p \to q) \to r$ is T is $\frac{5}{8}$.

**6**. Five ducks are swimming around the UNM duck pond which (for the purposes of this problem) has a circumference of 36 meters. The ducks begin a race around the whole circumference of the duck pond at the same time. Suppose that Daud the duck swims at 9 meters per minute, Dewi the duck swims at 6 meters per minute, Dino the duck swims at 4 meters per minute, and Duru the duck swims at 3 meters per minute.

Dyan the duck starts at 1 meter per minute, but Dyan's speed doubles every time another duck crosses the finish line. How many minutes does Dyan the duck take to finish the race?

**ANSWER:** 11

**Solution.** We thank Sean Choi for this problem. The given information determines the following table.

| duck | speed (meters/min) | time to finish (min) |
|---|---|---|
| Daud | 9 | $36/9 = 4$ |
| Dewi | 6 | $36/6 = 6$ |
| Dino | 4 | $36/4 = 9$ |
| Duru | 3 | $36/3 = 12$ |

*Provisionally*, Dyan's speed $v_{\text{Dyan}}$ is then the piecewise function

$$v_{\text{Dyan}} = \begin{cases} 1 & \text{for } 0 \leq t < 4 \\ 2 & \text{for } 4 \leq t < 6 \\ 4 & \text{for } 6 \leq t < 9 \\ 8 & \text{for } 9 \leq t < 12 \\ 16 & \text{for } t \geq 12. \end{cases}$$

We write *provisionally* here, since we have not yet determined how long Dyan takes to reach the finish line, and so if all of these velocities are attained. Dyan travels 4 meters over $0 \leq t < 4$, 4 meters over $4 \leq t < 6$, and 12 meters over $6 \leq t < 9$, for a total of 20 meters over $0 \leq t < 9$. Therefore, now with the speed 8 meters/min, she reaches the finish line after an additional 2 minutes, that is after 11 total minutes which puts her behind Dino, but ahead of Duru.

**Remark:** One student requested clarification, smartly asking if the ducks leave the race once crossing the finish line, or keep swimming. For example, does Daud leave the race at time $t = 4$, or keep swimming and cross the finish line again at $t = 8$. The problem statement implicitly assumes that the ducks leave the race after the first finish-line crossing, but we can also solve the problem assuming that they keep swimming. Under the "keep swimming" scenario, we have the following table.

| duck | speed | first crossing time | second crossing time | third crossing time |
|---|---|---|---|---|
| Daud | 9 | 4 | 8 | 12 |
| Dewi | 6 | 6 | 12 | 18 |
| Dino | 4 | 9 | 18 | 27 |
| Duru | 3 | 12 | 24 | 36 |

*Provisionally*, Dyan's speed is now the piecewise function

$$v_{\text{Dyan}} = \begin{cases} 1 & \text{for } 0 \leq t < 4 \\ 2 & \text{for } 4 \leq t < 6 \\ 4 & \text{for } 6 \leq t < 8 \\ 8 & \text{for } 8 \leq t < 9 \\ 16 & \text{for } 9 \leq t < 12. \end{cases}$$

Presumably, Dyan's speed would jump to 64 at $t = 12$, as both Daud and Dewi have subsequent crossings at this time. Now Dyan travels 4 meters over $0 \leq t < 4$, 4 meters over $4 \leq t < 6$, 8 meters over $6 \leq t < 8$, and 8 meters over $8 \leq t < 9$. Therefore, Dyan as travelled 24 meters by $t = 9$, when Dyan's speed again doubles to 16. Still 12 meters from the finish line, Dyan takes an additional $\frac{12}{16} = \frac{3}{4}$ minutes to reach it. Dyan then makes the first finish-line crossing at $t = 9\frac{3}{4}$ minutes. Given that the answer must be an integer per the instructions, we now know this interpretation of the problem is not possible.

**7**. Given two positive integers $a$ and $b$, their greatest common divisor $\texttt{gcd}(a, b)$ is the largest positive integer which divides them both. For example, $\texttt{gcd}(4, 21) = 1$, whereas $\texttt{gcd}(6, 21) = 3$. Find the product $ab$, assuming that $\texttt{gcd}(a, b) = 5$ and in the expansion of $(a + bx)^4$ it turns out that the coefficient of $x^2$ equals the coefficient of $x^3$.

**ANSWER:** 150

**Solution.** We adopt the direct approach.

$$(a + bx)^2 = a^2 + 2abx + b^2x^2,$$

therefore

$$\begin{aligned}(a + bx)^4 &= (a^2 + 2abx + b^2x^2)(a^2 + 2abx + b^2x^2)\\ &= a^4 + 2a^3bx + a^2b^2x^2\\ &+ 2a^3bx + 4a^2b^2x^2 + 2ab^3x^3\\ &+ b^2a^2x^2 + 2ab^3x^2 + b^4x^4\\ &= a^4 + 4a^3bx + 6a^2b^2x^2 + 4ab^3x^3 + b^4x^4.\end{aligned}$$

This expansion can be obtained more quickly with Pascal's triangle. We are told that $6a^2b^2 = 4ab^3$ or $3a = 2b$. Therefore, $a = 2\lambda$ and $b = 3\lambda$, but $\lambda$ must be 5 since $\texttt{gcd}(a, b) = 5$. So $a = 10$ and $b = 15$, and the product is 150.

**8**. A circular disk is divided by 362 equally spaced radii. What is the maximum number of non-overlapping areas into which the disk can be divided by the addition of a chord drawn across the bounding circle?

**ANSWER:** 544

**Solution 1.** To gain insight, we first consider the cases where the disk is divided by $2n = 2, 4, 6,$ and 8 equally spaced radii; see Fig. 2. Observe that these radii then come in pairs, with each pair forming a diagonal. The cases shown in the figure then correspond to $n = 1, 2, 3,$ and 4 diameters, respectively dividing the disk into $2n = 2, 4, 6,$ and 8 wedges of equal area. For each case in the figure, we have also drawn a red chord which intersects each of the diameters. Respectively, this results in $4, 7, 10,$ and 13 non-overlapping areas, and, by all appearances, this is the best we can achieve. In terms of the number $n$ of diameters, the pattern suggests that $3n + 1$ is the corresponding number of non-overlapping areas. Since 362 (equally spaced radii) corresponds to $n = 181$ (diameters), we conjecture that the number $3n + 1$ of non-overlapping areas is $3 \cdot 181 + 1 = 544$. **Note:** this "solution" is really an educated guess.

**Solution 2.** We have 362 equally spaced radii determining $A = 362$ areas (wedges, or pizza slices). To get a feel for the problem, we start with fewer radii. We start with an even number $2n$ of radii, and so $A = 2n$ wedges to begin with. The goal is to draw a chord $C$ such that the new number $A_C \geq A$ of areas is as large as possible. As a practical problem, we are making a straight cut on the pizza, and some slices are being split. We want to maximize the number of pieces obtained with just one straight cut, regardless of their size and shape. Given $n$, there is a finite number of configurations (all the chords of a certain type, enumerated below) which yield a distinct number $A_C$. Therefore, there will be a largest $A_C$ that we denote $A_n$.

When $n = 1$, the 2 equally spaced radii form a single diameter $d$ which divides the disk into two half-disks, and $A = 2$. The possible chords are as follows. (i) $C$ coincides with $d$; no new areas are created and $A_C = 2$. (ii) $C$ does not intersect $d$; $C$ will leave untouched one of the half-disks, and split the other in two, so $A_C = 3$. (iii) $C$ intersects $d$ at exactly one point, with two sub-cases: (iii.a) the intersection point is on the boundary, so $C$ splits one half-disk in two and $A_C = 3$, and (iii.b) the intersection point is interior, so $C$ splits both half-disks in two and $A_C = 4$. Therefore, $A_1 = 4$.

When $n = 2$, each of the 4 equally spaced radii will have an opposite partner, with the pair forming a diameter. So we have two diameters, $d_1$ and $d_2$, intersecting at a $90°$ angle and splitting the disk into 4 quarter-disks, so $A = 4$. The possible chords are as follows. (i) $C$ coincides with $d_1$ or $d_2$; no new areas are created and $A_C = 4$. (ii) $C$ intersects neither $d_1$ nor $d_2$; $C$ will leave untouched three of the quarter-disks, and split the remaining one in two, so $A_C = 5$. (iii) $C$ intersects either $d_1$ or $d_2$, but not the other. The intersection point may be on the boundary (iii.a), with $A_C = 5$, or in the interior (iii.b), with $A_C = 6$. (iv)

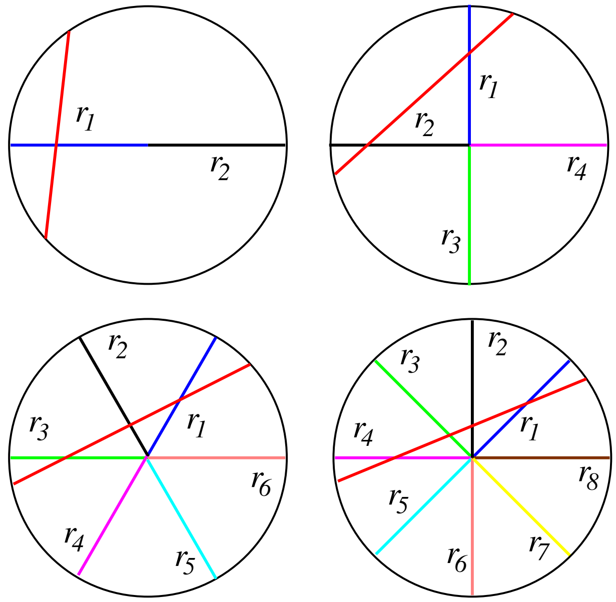



FIGURE 2

$C$ intersects both $d_1$ and $d_2$, with four sub-cases. (iv.a) $C$ is a distinct diameter intersecting both $d_1$ and $d_2$ at the center; so $C$ splits two opposite quarter-disks in two, and $A_C = 6$. (iv.b) $C$ intersects both $d_1$ and $d_2$ at boundary points, with $A_C = 5$. (iv.c) $C$ intersects either $d_1$ or $d_2$ at a boundary point, but the other diameter in the interior, so that $A_C = 6$. (iv.d) $C$ intersects both $d_1$ and $d_2$ at distinct interior points. In fact, these two intersection points must lie on consecutive radii that border three quarter-disks. Each of these three quarter-disks is split by $C$, adding three areas, so $A_C = 7$. Therefore, $A_2 = 7$.

When $n = 3$, the 6 radii will correspond to 3 diameters $d_1$, $d_2$, and $d_3$, with any two intersecting at a 60° angle. Now we start with sixth-disks, and $A = 6$. The possible chords are as follows. (i) $C$ coincides with with one of the diameters; $A_C = 6$. (ii) $C$ intersects none of the diameters; $A_C = 7$. (iii) $C$ intersects either $d_1$, $d_2$, or $d_3$, but not the other two. The intersection point may be on the boundary (iii.a), with $A_C = 7$, or in the interior (iii.b), with $A_C = 8$. (iv) $C$ intersects precisely two of $d_1$, $d_2$, and $d_3$, but not the remaining one, with three sub-cases. (iv.a) The two intersection points are on the boundary; $A_C = 7$. (iv.b) One intersection point is on the boundary, the other in the interior; $A_C = 8$. (iv.c) Both intersection points are interior; $A_C = 9$. (v) $C$ intersects $d_1$, $d_2$, and $d_3$. Sub-cases abound. (v.a) The center is the common intersection point, with $C$ a distinct diameter; $A_C = 8$. (v.b) Two intersection points lie on the boundary, with one interior; $A_C = 8$. (v.c) One intersection point lies on the boundary, with two interior; $A_C = 9$. (v.d) All three intersection points are interior. $C$ intersects the three diameters at distinct points. The three intersection points must lie on consecutive radii that border four sixth-disks. Each of these four sixth-disks is split by $C$, adding four areas, so $A_C = 10$. Therefore, $A_3 = 10$.

A pattern emerges. The given $2n$ equally spaced radii correspond to $n$ diameters, and our experimentation with fewer radii indicates the following. The largest number of created areas arises when the chord $C$ is not a diameter and intersects all $n$ diameters at $n$ distinct interior points. These points lie on $n$ consecutive radii that border $n+1$ areas. These $n+1$ areas are split by $C$, and $n+1$ areas will be added to the total; that is $A_n = A + n + 1 = 3n + 1 = \frac{3}{2}A + 1$. In particular, if $A = 362$, then $n = 181$ and $A_{181} = 544$.

**Solution 3.** The number 362 is large enough that we choose to generalize the problem. Assume an arbitrary even number $2n$ of equally spaced radii $\{r_1, r_2, \ldots, r_{2n}\}$, emanating from the origin at angles $\theta_k = 2k\pi/(2n)$, where $k = 1, 2, \cdots, 2n$. With the restriction $1 \le k \le n$, we have

$$\theta_{k+n} = 2(k+n)\pi/(2n) = \pi + \theta_k.$$

So our division of the disk by $2n$ radii is in fact a division by $n$ diameters. If we choose our chord to be one of these diameters, then we add no new areas, instead keeping the $2n$ wedges we started with. If we choose

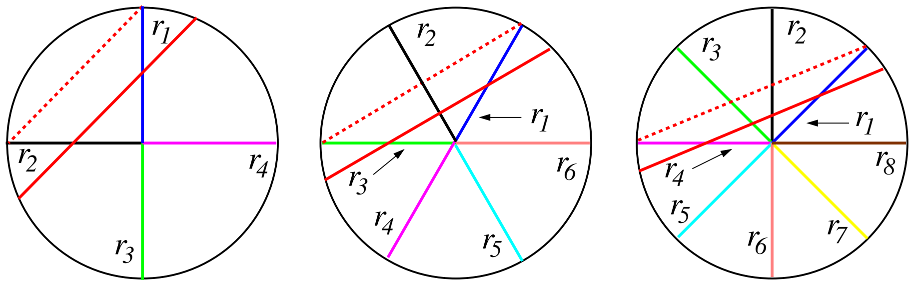



FIGURE 3

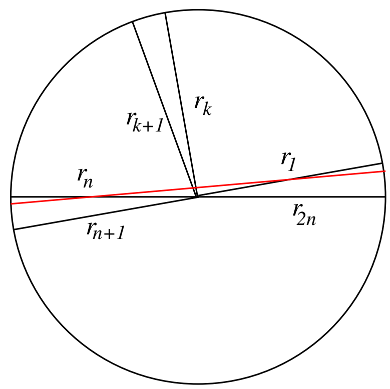



FIGURE 4

a different diameter, then it will be nestled between two of the consecutive given diameters. Then the new diameter will split two wedges, adding just two new areas for a total of $2n + 2$ non-overlapping areas. We can clearly do better by choosing a chord which is not a diameter.

One possibility is the chord which connects the tips (endpoints on the circular boundary) of $r_1$ and $r_n$. Figure 3 shows this particular chord as a dotted red segment for $n = 2, 3, 4$. This chord will divide $n - 1$ wedges in two. As we start with $2n$ wedges, splitting $n - 1$ of these results in $3n - 1$ non-overlapping areas. Not bad, but can we do better?

An optimal chord will cut across *all* $n$ diameters. Such a chord may arise by perturbing the chord which connects the tips of $r_1$ and $r_n$. Indeed, if this tip-connecting chord is parellel transported (with simultaneous extension so that it remains a chord) towards the center, then a new chord[1] arises which indeed cuts across all diameters. Figure 3 depicts the new perturbed chord as a solid red segment for the cases $n = 2, 3, 4$. The perturbed chord still divides the same $n - 1$ wedges as the tip-connecting chord did, but now also divides two further wedges. Indeed, consider the perturbed chord as depicted by a red segment in Fig. 4. Let $A$, $P$, $Q$, $B$ be points on the red chord, labeled left to right, corresponding to lying on the circle, at the intersection

[1]We can write down the equation for such a chord. If our disk has radius $R$, then the tips of $r_1$ and $r_n$ are $R(\cos\theta_1, \sin\theta_1)$ and $R(\cos\theta_n, \sin\theta_n)$, respectively. So the tip-connecting chord has the point-slope equation

$$\frac{y - R\sin(\theta_1)}{x - R\cos\theta_1} = m, \qquad m = \frac{\sin\theta_n - \sin\theta_1}{\cos\theta_n - \cos\theta_1} = \frac{\sin\theta_1}{1 + \cos\theta_1} = \tan(\tfrac{1}{2}\theta_1),$$

where we have used a half-angle identity to simplify the expression for the slope $m$. In slope-intercept form this equation is

$$y = mx + b, \qquad b = R(\sin\theta_1 - m\cos\theta_1).$$

One possible choice for an optimal chord is then

$$y = mx + \tfrac{1}{2}b.$$

This line is parallel to the tip-connecting line, but shifted toward the center of the disk. To find the $x$ values which correspond to where this line intersects the boundary of the disk (the chord endpoints), we need to solve a nonlinear system of equations (the above equation along with $x^2 + y^2 = R^2$). This would amount to finding the roots of $(1 + m^2)x^2 + mbx + \frac{1}{4}b^2 - R^2 = 0$.

with $r_n$, at the intersection with $r_1$, and on the circle. Then the segment $PQ$ cuts $n-1$ wedges precisely in two, giving $2n-2$ non-overlapping areas. The segments $AP$ and $QB$ each subdivide a wedge in two, giving 4 non-overlapping areas. Finally, we have $n-1$ wedges left over which are not subdivided by the added chord. In all we find $(2n-2)+4+(n-1) = 3n+1$ non-overlapping areas. We should remark on why we cannot do better. The described optimal chord starts in one wedge and ends in the antipodal wedge, while passing between all wedges which lie to one side and between these two wedges. A *straight* segment cannot pass through yet another wedge in addition to these. If the red line did pass into yet another wedge, it would have crossed a diagonal at two distinct points, and that is not possible.

Our analysis of the generalized problem determines that at most 544 non-overlapping areas arise in drawing a chord across a disk subdivided by $2n = 362$ equally spaced radii.

**9**. If $R$ is expressed in lowest terms, what is the denominator of $R$?

$$R = \frac{1+3+5+7+\cdots+999}{2+4+6+\cdots+1000}$$

**ANSWER:** $\boxed{501}$

**Solution 1.** Following the footsteps[2] of Johann Carl Friedrich Gauss as a child, we know that the sum of the first $N$ natural numbers is

$$1+2+\cdots+N = \tfrac{1}{2}N(N+1). \tag{1}$$

Gauss' idea can be exploited here. First, since 1000 is even, the numbers between and including 1 and 1000 can be split into 500 odd and 500 even numbers. For these odd numbers, we consider

$$\begin{array}{rcccccccccc} \text{numerator} & = & 1 & + & 3 & + & \cdots & + & 997 & + & 999 \\ \text{numerator} & = & 999 & + & 997 & + & \cdots & + & 3 & + & 1 \\ \hline 2(\text{numerator}) & = & 1000 & + & 1000 & + & \cdots & + & 1000 & + & 1000 \end{array}$$

Since there are 500 terms in these sums, $2(\text{numerator}) = 500 \cdot 1000$, so that numerator $= 500^2$. Likewise, for these even numbers, we consider

$$\begin{array}{rcccccccccc} \text{denominator} & = & 2 & + & 4 & + & \cdots & + & 998 & + & 1000 \\ \text{denominator} & = & 1000 & + & 998 & + & \cdots & + & 4 & + & 2 \\ \hline 2(\text{denominator}) & = & 1002 & + & 1002 & + & \cdots & + & 1002 & + & 1002 \end{array}$$

We then find $2(\text{denominator}) = 500 \cdot 1002$, so that denominator $= 500 \cdot 501$. So $R$ can be expressed as

$$R = \frac{500}{501},$$

a result evidently in lowest terms.

**Solution 2.** Let $P$ denote the sum of odd numbers less than 1000, and $Q$ the sum of even numbers less than or equal to a 1000,

$$P = 1+3+5+\cdots+997+999, \qquad Q = 2+4+6+\cdots+998+1000.$$

Then $R = P/Q$. By the Gauss formula (1) with $N = 1000$,

$$P+Q = 1+2+3+\cdots+999+1000 = \frac{1000 \times 1001}{2} = 500 \times 1001. \tag{2}$$

[2]According to lore, when asked to add the first 100 natural numbers ($N = 100$) a seven-year old Gauss wrote the following in his mind.

$$\begin{array}{ccccccccccc} S & = & 1 & + & 2 & + & \cdots & + & N-1 & + & N \\ S & = & N & + & N-1 & + & \cdots & + & 2 & + & 1 \\ \hline 2S & = & (N+1) & + & (N+1) & + & \cdots & + & (N+1) & + & (N+1) \end{array}$$

So $S = \frac{1}{2}N(N+1)$. Gauss then concluded that sum of the first hundred natural numbers was $5050 = \frac{1}{2} \times 100 \times 101$.

After pulling out and overall factor of 2 in $Q$, we use the same formula with $N = 500$ to find

$$Q = 2(1 + 2 + \cdots + 500) = 2\frac{500 \times 501}{2} = 500 \times 501. \tag{3}$$

With (2) and (3), we solve for $P$, finding

$$P = 500 \times 1001 - Q = 500 \times 1001 - 500 \times 501 = 500 \times (1001 - 501) = 500^2.$$

Therefore,

$$R = \frac{P}{Q} = \frac{500^2}{500 \times 501} = \frac{500}{501}.$$

Since $500 = 5 \times 100 = 5^3 \times 2^2$ and 501 is not divisible by either 2 or 5, then the fraction $\frac{500}{501}$ is in lowest terms. Its denominator is 501.

**Note:** As an exercise, you can verify that if $P_N$ denotes the sum of the odd numbers less than or equal to $N$, and $Q_N$ the sum of the even numbers less than or equal to $N$, then, if $N$ is an even number,

$$P_N = (\tfrac{1}{2}N)^2, \qquad Q_N = (\tfrac{1}{2}N)(\tfrac{1}{2}N + 1).$$

What if $N$ is an odd number?

**Solution 3.** One can look for a pattern, by considering other even numbers $n = 2k$, and analyzing the fraction

$$R_k = \frac{1 + 3 + \cdots + 2k - 1}{2 + 4 + \cdots + 2k}.$$

For example, when $n = 2$, $k = 1$ and $R_1 = \frac{1}{2}$. When $n = 4$, $k = 2$ and $R_2 = \frac{1+3}{2+4} = \frac{4}{6} = \frac{2}{3}$. Shall we try one more? When $n = 6$, $k = 3$ and $R_3 = \frac{1+3+5}{2+4+6} = \frac{9}{12} = \frac{3}{4}$. At this point, we might guess that

$$R_k = \frac{k}{k+1}. \tag{4}$$

If valid, then for $n = 1000$ we have $k = 500$ and $R_{500} = \frac{500}{501}$, the correct answer. So far, the last formula is a guess; we will verify our guess by induction on $k$. We have already checked the base case(s). Assuming (4), we want to show that $R_{k+1} = \frac{k+1}{k+2}$ follows. We start with

$$R_{k+1} = \frac{1 + 3 + \cdots + (2(k+1) - 1)}{2 + 4 + \cdots + 2(k+1)} = \frac{1 + 3 + \cdots + (2k-1) + (2k+1)}{2 + 4 + \cdots + 2k + 2(k+1)},$$

where the first equality is just the definition, and the second follows by including the second to last term in the numerator and denominator, also noting that $2(k+1) - 1 = 2k + 1$. With last equation and some algebra, we then find

$$\begin{aligned} R_{k+1} &= \frac{1 + 3 + \cdots + (2k-1)}{2 + 4 + \cdots + 2k + 2(k+1)} + \frac{2k+1}{2 + 4 + \cdots + 2k + 2(k+1)} \\ &= \left(\frac{1 + 3 + \cdots + (2k-1)}{2 + 4 + \cdots + 2k}\right)\left(\frac{2 + 4 + \cdots + 2k}{2 + 4 + \cdots + 2k + 2(k+1)}\right) + \frac{2k+1}{2 + 4 + \cdots + 2k + 2(k+1)}. \end{aligned} \tag{5}$$

To obtain the first equality, we have used $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$, with $a = 1 + 3 + \cdots + (2k-1)$, $b = 2k + 1$, and $c = 2 + 4 + \cdots + 2k + 2(k+1)$ the common denominator. To obtain the second equality, we multiplied and divided the first term $\frac{a}{c}$ on the right-hand side by $d = 2 + 4 + \cdots + 2k$, followed by the reorganization $\frac{a}{c}\frac{d}{d} = \frac{a}{d}\frac{d}{c}$. As a result of these steps, we have introduced the fraction

$$\frac{a}{d} = \frac{1 + 3 + \cdots + (2k-1)}{2 + 4 + \cdots + 2k},$$

precisely the expression for $R_k$, which we are assuming is equal to $\frac{k}{k+1}$. With this observation, we write the last expression in (5) as

$$R_{k+1} = \frac{k}{k+1}\left(\frac{2 + 4 + \cdots + 2k}{2 + 4 + \cdots + 2k + 2(k+1)}\right) + \frac{2k+1}{2 + 4 + \cdots + 2k + 2(k+1)}. \tag{6}$$

Next, we make two appeals to Gauss' formula,

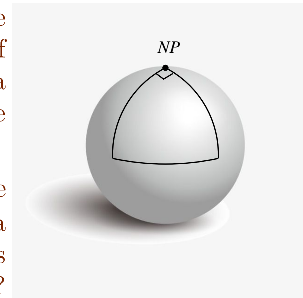
$$1 + 2 + \cdots + k = \frac{1}{2}k(k+1), \qquad 1 + 2 + \cdots + k + (k+1) = \frac{1}{2}(k+1)(k+2),$$

in order to further simplify (6), thereby reaching

$$R_{k+1} = \frac{k}{k+1}\frac{k(k+1)}{(k+1)(k+2)} + \frac{2k+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

By the Principle of Mathematical Induction, we have shown that $R_k = \frac{k}{k+1}$ for all $k \geq 1$. In particular, for $k = 500$ we have that $R_{500} = \frac{500}{501}$.

**10**. Relative to the center of the unit circle, a *radian* can be defined as the angle subtended by a unit-length arc on the unit circle. Likewise, relative to the center of the unit sphere, a *steradian* can be defined as the solid angle subtended by a unit-area portion (of any shape) on the unit sphere. The solid angle subtended by the unit sphere itself is $4\pi$ steradians.




Consider a round sphere, with radius $r = \sqrt{\frac{8\sqrt{3}}{\pi}}$, and two great circles which emanate from its north pole *NP*. At *NP* they intersect at a right angle, and together with a portion of the equator form an equilateral spherical triangle; see the figure. What is the side length of an equilateral triangle with the same area as this spherical triangle?

**ANSWER:** $\boxed{4}$

**Solution 1.** The preamble of the problem tell us that the area of a unit-radius sphere is $4\pi$, and so by dimensional analysis the area of a radius-$r$ sphere is $4\pi r^2$. The area of the described spherical triangle is $\frac{1}{8}$ the area of the described sphere, namely

$$A_{\text{spherical}\triangle} = \tfrac{1}{8}\cdot 4\pi r^2 = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi\big(8\sqrt{3}/\pi\big) = 4\sqrt{3}.$$

Therefore, if the side length of the planar triangle is $a$, then we require

$$A_{\text{planar}\triangle} = \tfrac{1}{4}\sqrt{3}a^2 = 4\sqrt{3} \implies a^2 = 16,$$

and $a = 4$. Here we have used the formula for the area of a planar equilateral triangle. To recall how that formula arises, let $A_{\text{planar}\triangle} = \frac{1}{2}ah$, where $h$ is the height of the equilateral triangle with side $a$. Applying the *Pythagorean Theorem* to the right triangle with hypotenuse $a$ and sides $\frac{1}{2}a$ and $h$,

$$h^2 + \tfrac{1}{4}a^2 = a^2 \qquad \iff \qquad h^2 = \tfrac{3}{4}a^2,$$

we conclude that $h = \frac{1}{2}\sqrt{3}a$ and $A_{\text{planar}\triangle} = \frac{1}{4}\sqrt{3}a^2$.

**Solution 2.** This "solution" is really an educated guess relying on all available knowledge; in particular, that the answer should be an integer. Say we can't make heads nor tails of the statement in the problem's preamble. The depicted equilateral spherical triangle has sides (in fact arcs) of length

$$\begin{aligned}\ell &= \Big[\text{one fourth of the circumference of a circle with radius } r = \sqrt{8\sqrt{3}/\pi}\Big]\\ &= \tfrac{1}{2}\pi\sqrt{8\sqrt{3}/\pi}\\ &= \pi\sqrt{2}\times\sqrt{\sqrt{3}/\pi}\\ &= \pi\times\sqrt{\sqrt{4/3}}\times\sqrt{3/\pi}.\end{aligned}$$

Remember, we don't have a calculator, so we will have to cut some corners to get a decimal number from the last expression. Both square-root factors here should be numbers reasonably close to 1. Indeed,[3] $4/3$ is close to 1, $\sqrt{4/3}$ is closer still, and $\sqrt{\sqrt{4/3}}$ is even closer. Likewise, $3/\pi$ is close to 1, and $\sqrt{3/\pi}$ is closer still.

[3] You may be aware that for any number $x > 0, x \neq 1$ the number $\sqrt{x}$ is closer to 1 than $x$ is itself. The following calculation justifies this assertion.

$$|1-\sqrt{x}| = \left|\frac{(1-\sqrt{x})(1+\sqrt{x})}{1+\sqrt{x}}\right| = \frac{|1-x|}{1+\sqrt{x}} < |1-x|$$

The final strict inequality holds because $|1-x| \neq 0$ and $1+\sqrt{x} > 1$, so that $0 < 1/(1+\sqrt{x}) < 1$.

We conclude that the side length $\ell$ of our spherical triangle is a number close to[4] $\pi = 3.14159\cdots$. Since we are told the answer is an integer, the corresponding equilateral planar triangle likely has side length 3 or 4 (the closest integers to $\pi$). But which one? For the same side length, we expect that an equilateral spherical triangle will have a *larger* area than an equilateral planar triangle. Why? Well, the spherical triangle bulges out in the middle. Therefore, to have the same area as the spherical triangle, the planar triangle in question should have a *longer* side length. Our educated guess is then 4.

**Remark.** The problem relies on a spherical triangle, a region on the sphere bounded by three arcs of great circles, and the notion of a right angle for a spherical triangle. In fact, for the shown equilateral spherical triangle, the angle at each vertex is right, and these three angles sum to $\frac{3}{2}\pi$ (or 270°). So this is definitely not a *planar* triangle! While we have assumed that the problem can be solved relying on an intuitive notion of such a right angle, one way to understand angles on the sphere is through *stereographic projection.*[5]

Let $P$, with Cartesian coordinates $(x, y, z) = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, be a point on the radius-$r$ sphere $x^2+y^2+z^2 = r^2$. Respectively, $\theta$ and $\phi$ are the polar and azimuthal angles,[6] with $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. The equator of the sphere corresponds to $\theta = \frac{1}{2}\pi$. Given the point $P$ on the sphere, we define a corresponding point $Q$ on the $z = 0$ plane, also called the $xy$-plane. We choose $Q$ such that $Q$, $P$, and the South Pole $SP = r(0, 0, -1)$ are colinear, although $NP = r(0, 0, 1)$ often replaces $SP$ in the construction. We skip the derivation, but this choice determines that $Q$ has coordinates $(x, y, z) = r\tan(\frac{1}{2}\theta)(\cos\phi, \sin\phi, 0)$. This transformation defines a stereographic projection mapping the sphere to the plane. Great circles of longitude on the sphere correspond to fixation of the azimuthal angle $\phi$; they run from the North Pole ($\theta = 0$) through the South Pole ($\theta = \pi$). These circles are mapped to straight lines through the origin $O$ in the $xy$-plane. The North Pole is mapped to $O$, and, in a sense, the South Pole to $\infty$. Figure 5 shows other images.

Where do angles enter the picture? Given two curves (say two arcs of great circles) on the sphere which intersect at a point $P$ (assumed *not* $SP$), we could define the angle at $P$ between these spherical curves as follows. First, use the stereographic projection to map the spherical curves to planar curves in the $xy$-plane which intersect at the point $Q$ corresponding to $P$. Next, compute the angle at $Q$ between the planar curves in the usual sense, Take this angle to define the angle at $P$ between the spherical curves. With this definition, the angle at $NP$ between two great circles is the angle between two straight rays in the plane emanating from $O$. The equilateral spherical triangle in the problem statement then has right angles at all vertices, because all vertex angles in its image under the stereographic projection are right; again see Fig. 5.

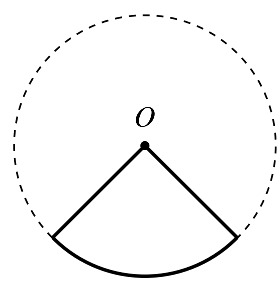



Figure 5. Image of the equator (dotted circle) and image of the equilateral spherical triangle (solid "pizza slice") under stereographic projection. The disk within the circle is the image of the Northern Hemisphere.

[4] Although we would not be able to check in the live test, the estimate $\ell \simeq \pi$ is indeed close to the exact value $\ell = 3.29890\cdots$. It's only off by about 5 percent in a relative sense. While sneaking some calculator use in this footnote, let us also note that $\sqrt{\sqrt{4/3}} = 1.07456\cdots$ and $\sqrt{3/\pi} = 0.97720\cdots$. The product $\sqrt{\sqrt{4/3}} \times \sqrt{3/\pi} = 1.05007\cdots$, and $\ell = \pi(1.05007\cdots)$.

[5] According to Wikipedia, stereographic projection was likely known by ancient Greek astronomers who used it to map the *celestial sphere* onto the plane. The trajectories of the stars and planets could then be analyzed using planar geometry.

[6] These angles are standard in physics and engineering, but mathematicians often interchange the role of $\theta$ and $\phi$.