UNM-PNM Statewide High School Mathematics Contest LVII
Round-2 Solutions with Comments

Dear Students,
If you have suggestions about the Contest, or if you have a different solution to any of this year's second-round problems, please mail them to:

Professors Stephen Lau and Cristina Pereyra
Department of Mathematics and Statistics
1 University of New Mexico
MSC01 1115
Albuquerque, NM 87131

or email them to: `lau@math.unm.edu` and `crisp@math.unm.edu`. Please remember that you can find information about past contests at `http://mathcontest.unm.edu`

We express our gratitude to Coach Sean Choi for the time spent with students in Albuquerque and Los Alamos, and for running online sessions to review problems from the first and second rounds. We also thank Sean and Bill Cordwell for sharing their solutions with us. As you will see, we drew on students' work for many of the solutions presented below. Finally, thanks to all participants, their teachers, and families. You are an inspiration for us!

---

1. A two-pan balance scale is used to identify a counterfeit coin via weight comparisons.

(a) Nine coins are identical in appearance, but one is counterfeit. The counterfeit coin is heavier than the others. How can you guarantee identification of the counterfeit coin with at most two scale comparisons?

(b) Find the least number of scale comparisons necessary to guarantee identification of one heavy counterfeit coin among 27 coins which are identical in appearance.

Preface. Some students said they would first use their hands (a two-pan balance of sorts!) to compare coins until the heavier counterfeit coin had been identified, and then confirm the identification with the two-pan balance. This proposal assumes that a person is able to discern the weight difference between a genuine coin and the counterfeit. However, if the weight difference is small enough, a person, no matter how well-trained, will not be able to discern the difference. Implicit in the problem statement is the assumption that our two-pan balance can discern the weight difference. No matter how small the difference, upon comparison of two groups of coins (each group with the same number of coins, with one group containing the counterfeit), the two-pan balance will perceptibly tilt.

Other students assumed that the counterfeit was twice as heavy as the other coins. Would such an additional piece of information, not stated in the problem, lead to a different comparison strategy than the ones described below?

In presenting your arguments, many of you sketched some splendid diagrams!

Solution, based in part on an approach given by Inler Lu, 8th grade, Desert Ridge Middle School. For **(a)** our strategy will produce a sequence S_2, S_1, S_0 of sets, where S_k holds 3^k coins, with the counterfeit always among them. The singleton set S_0 will then have the counterfeit coin as its only element. First, let S_2 hold the initial 9 coins, and then randomly divide S_2 into three subsets A_1, B_1, C_1, where each subset holds 3 coins. Using the scale, compare the weights of A_1 and B_1. Either (i) A_1 and B_1 weigh the same, and we know the fake coin is in C_1, or (ii) one of A_1 and B_1 weighs more. Say A_1 weighs more. Then we know that the fake coin is in A_1. Either way, we have determined that the fake coin lies within a subset S_1 (either A_1, B_1, or C_1) of 3 coins. Now randomly divide S_1 into three singleton sets A_0, B_0, C_0, where each holds a single coin. A second use of the scale comparing A_0 against B_0 identifies the fake coin. Indeed, if A_0 and B_0 have equal weight, then C_0 holds the counterfeit. Otherwise, the heavier of A_0 and B_0 holds the counterfeit. For convenience below, we let S_0 (either A_0, B_0, or C_0) be the final singleton set holding the counterfeit coin.

For **(b)** let S_3 be the initial set of 27 coins, and randomly divide this set into three subsets A_2, B_2, C_2, where each subset holds 9 coins. Thus, similar to **(a)**, we start with three sets, although now each set has 9 coins. A single use of the scale, comparing the weights of A_2 and B_2, will then identify which of these three sets contains the counterfeit; let this set be S_2. Now we have a set S_2 of 9 coins containing the fake coin, and so are precisely back to the scenario in part **(a)**. Therefore, after the first scale comparison needed to identify S_2, another two comparisons will be needed to get down to S_0, the singleton set holding the fake. Therefore, three scale comparisons will ensure identification of the counterfeit coin.

We still need to justify that, starting with the 27-coin scenario, 3 is the *least* number of scale comparisons needed to identify the counterfeit. Evidently, a starting collection of 2 or more coins will require at least one scale comparison to identify the fake coin. Perhaps not as obvious, if we start with between 4 and 9 coins, then we will need at least 2 comparisons. Why? For a first scale comparison, we will need to divide the starting collection into groups A, B, and C, where A and B are non-empty and have the same number of coins, and C might be empty (have no coins). Think of this initial division as preparing sets A and B for the first comparison. At least one of the groups A, B, C will have 2 or more coins. For example, with a 4-coin starting collection, the possible splits into A, B, C are 2-2-0 and 1-1-2. In the first case, we compare and keep the heaviest of the 2-coin groups (the A and B groups), but would then need one more comparison to identify the counterfeit. In the second case, we compare the two single-coin groups (again, the A and B groups), and might get lucky and find the counterfeit, should one of these groups prove heavier than the other. However, should these two single-coin groups have the same weight, then we will need a second comparison to identify the counterfeit among the leftover 2-coin group (the C group). If we start with a collection of between 10 and 27 coins, then we will need at least 3 comparisons. Why? To make an initial comparison and learn something, we again need to divide the collection into groups A, B, and C, where A and B are non-empty and have the same number of coins, and C might be empty. This initial division will always result in a group with 4 or more coins. For example, with 10 coins this initial division could be 5-5-0, 4-4-2, 3-3-4, 2-2-6, 1-1-8. After the first comparison of A with B, the counterfeit coin could be in a group (either A, B, or C) that has 4 or more coins; to identify it from that group, we will need at least 2 more comparisons. This argument shows that identification of the fake coin requires a minimum of 3 comparisons, when starting with between 10 and 27 coins.

Algorithm 1 IDENTIFICATION OF A HEAVY FAKE COIN AMONG A SET OF $N = 3^p$ COINS. We assume $p = \log_3 N \geq 1$ is an integer. On exit S_0 is the singleton set holding the fake.

Let S_p be the initial collection of $N = 3^p$ coins.
for $k = p$ **down to** 1
    Divide S_k into subsets $A_{k-1}, B_{k-1}, C_{k-1}$, each with 3^{k-1} coins.
    if $A_{k-1} = B_{k-1}$ ▷ Compare the weights of A_{k-1} and B_{k-1}.
        Set S_{k-1} equal to C_{k-1}.
    else
        if $A_{k-1} > B_{k-1}$
            Set S_{k-1} equal to A_{k-1}.
        else
            Set S_{k-1} equal to B_{k-1}.
        end
    end
end

Comments.

• Our strategy works for a collection of 3^p coins, with a single heavy fake coin among them. The process is given in **Algorithm 1**, and it requires p comparisons. AKILAN SANKARAN, 12th grade, Albuquerque Academy, described this algorithm.

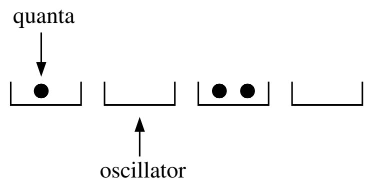
• Many students presented a natural strategy that we call *divide and conquer*. You choose from the starting collection two equal-number groups which are as large as possible, leaving no coin out (when you have an even number of coins) or just one coin out (when you have an odd number of coins). This strategy allows one to get lucky; it might identify the counterfeit coin after the first comparison, at least when you have an odd number of coins. The downside is that you may also be unlucky, needing more comparisons than the minimum necessary.

For **(a)**, divide the 9 coins into two 4-coin groups, with 1 coin left out. Compare the two 4-coin groups, if they a weigh the same, then the counterfeit coin is the one left out, and you are done! You were lucky!! However, if one of the 4-coin groups proves heavier than the other, then you know the counterfeit coin is in that group. Subdivide those 4 coins in two 2-coin groups, and compare them. One group will be heavier than the other, and you now know that the counterfeit coin is in that 2-coin group. You then need one more comparison to identify the counterfeit, for a total of 3 comparisons.

For **(b)**, divide the 27 coins into two 13-coin groups, with 1 coin left out. Compare the two 13-coin groups; if they weigh the same, the counterfeit coin is the coin left out, and you are done! You were lucky, and found the counterfeit coin with just one comparison!! However, if one of the 13-coin groups proves heavier than the other, then you know the counterfeit coin is in that group. Subdivide that 13-coin group into two 6-coin groups, with 1 coin left out. Compare the two 6-coin groups; if they weigh the same, then counterfeit coin is the coin left out, and you are done! You were lucky, and found the counterfeit coin with two comparisons!! If one of the 6-coin groups proves heavier than the other, then you know that the counterfeit coin is in the heavier 6-coin group. Subdivide that 6-coin group in two 3-coin groups, and compare. The heavier 3-coin group has the counterfeit coin. You need one more comparison to identify the counterfeit coin, for a total of 4 comparisons.

Divide-and-conquer is not optimal; it does not guarantee identification of the counterfeit coin with 2 (for 9 coins) or 3 (for 27 coins) comparisons.

2. Suppose M quanta of energy are distributed in N one-dimensional quantum mechanical oscillators. A key question (in quantum statistical mechanics!) is how many possible ways can the indistinguishable quanta be distributed. What is the answer for the scenario $M = 3$ and $N = 4$, with one possible state shown in the figure?



Solution 1. Dusty Dixon, 7th grade, Desert Ridge Middle School, assessed this problem as follows: "Those fancy words are to throw you off. Think of it as a simple question like you have 3 apples and 4 baskets." Exactly right! Many students listed all possibilities correctly and counted. Most organized the search as follows:

(i) All three quanta are in one oscillator. There are 4 oscillators, hence 4 configurations.

(ii) Two quanta are in one oscillator, the other quanta in another oscillator. The two quanta can be in any of the 4 oscillators, and the third quanta is then left to choose among the 3 remaining oscillators, for a total of $3 \times 4 = 12$ configurations.

(iii) Each of the three quanta are in three different oscillators. There is one empty oscillator that can be any of the 4 oscillators. So there are 4 configurations.

Respectively, these cases correspond to configurations in which the number of empty oscillators is 3, 2, and 1. Figure 1 enumerates all possible configurations. The leftmost grouping shows the 4 configurations in which all three quanta are placed in a single oscillator. The middle grouping shows the 12 configurations in which two quanta are placed in one oscillator and one quanta is placed in another oscillator. Finally, the rightmost grouping shows the 4 configurations in which each oscillator holds at most one quanta. The count is 20.

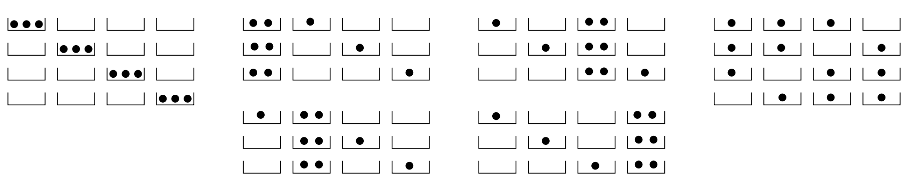


Figure 1. Possible configurations.

Solution 2. Some contestants referred to this approach as "stars and bars" or "sticks and stones". We use "quanta (q) and walls (w)". We keep track of the oscillators by the "walls" between them. Then, the configuration shown in the problem statement may be represented as $\{qwwqqw\}$. Since there are $N = 4$ oscillators, there are $N - 1 = 3$ walls. If all of the quanta and walls were distinguishable, we would have $M + N - 1 = 3 + 4 - 1 = 6$ distinguishable objects, and the total number of arrangements would be 6!. However, since the quanta are in fact indistinguishable from each other, and so are the walls, we divide both by the $3! = 6$ permutations of the quanta and the $(4 - 1)! = 3! = 6$ permutations of the walls. Hence the distinct number of configurations is

$$\frac{6!}{3!3!} = 20.$$

This approach also affords the general solution. Indeed, with N and M unspecified,

$$\frac{(M+N-1)!}{M!(N-1)!}$$

is the number of allowable configurations.

Solution 3, due to Owen Petersen, 11th grade, ASK Academy. Here we work inductively on the number of oscillators.

• For $N = 1$ oscillator, all M quanta must be in that oscillator, so 1 configuration.

• For $N = 2$ oscillators, we can have M_1 quanta in the first oscillator and M_2 quanta in the second, subject to $M_1 + M_2 = M$. The number M_2 is determined by M_1 and the fixed value of M. Moreover, M_1 can take any value between 0 and M; there are $M + 1$ configurations.

• For $N = 3$ oscillators, we can have M_1, M_2, and M_3 quanta in the first, second, and third oscillators, respectively, where $M_1+M_2+M_3 = M$. We can determine M_3 from M_1+M_2 and the fixed value of M. Let $k = M_1 + M_2$. Then $0 \le k \le M$, and, by the reasoning in the last bullet point, for each possible k there are $k+1$ ways the k quanta can be distributed among the first and second oscillators. Therefore, the total number of allowable configurations is

$$\sum_{k=0}^{M}(k+1) = \tfrac{1}{2}(M+1)(M+2).$$

This is 10 when $M = 3$, as we can confirm.

(1) $k = 0$. Both first and second oscillators are empty, third oscillator has 3.

$$(0, 0, 3)$$

(2) $k = 1$. First and second oscillators have total of 1, third oscillator has 2.

$$(1, 0, 2), \qquad (0, 1, 2)$$

(3) $k = 2$. First and second oscillators have total of 2, third oscillator has 1.

$$(1, 1, 1), \qquad (2, 0, 1), \qquad (0, 2, 1)$$

(4) $k = 3$. First and second oscillators have total of 3, third oscillator is empty.

$$(3, 0, 0), \qquad (0, 3, 0), \qquad (2, 1, 0), \qquad (1, 2, 0)$$

• For $N = 4$ oscillators, we can have M_1, M_2, M_3, and M_4 quanta in the first, second, third, and fourth oscillators, respectively, where $M_1 + M_2 + M_3 + M_4 = M$. Now we can determine M_4 from M_1, M_2, M_3 and the fixed value of M. Let ℓ be a possible value for $M_1 + M_2 + M_3$. Then by the reasoning of the last bullet, there are $\frac{1}{2}(\ell+1)(\ell+2)$ ways that the ℓ quanta can be distributed into the first three oscillators, where the remaining $M - \ell$ quanta must reside in the fourth oscillator. Since $0 \le \ell \le M$, the total number of allowable configurations is

$$\tfrac{1}{2}\sum_{\ell=0}^{M}(\ell+1)(\ell+2).$$

Using arguments based on telescoping series or induction (neither given here), one finds

$$\tfrac{1}{2}\sum_{\ell=0}^{M}(\ell+1)(\ell+2) = \tfrac{1}{6}(M+1)(M+2)(M+3).$$

However, this general expression is not needed here. Indeed, for $M = 3$ we clearly have

$$\tfrac{1}{2}\sum_{\ell=0}^{3}(\ell+1)(\ell+2) = \tfrac{1}{2}\big(2+6+12+20\big) = 20.$$

This approach can be used to establish the result

$$\binom{M+N-1}{M} = \frac{(M+N-1)!}{M!(N-1)!}$$

for arbitrary N and M found above by the "stars and bars" method. Indeed, this expression gives 1, $M{+}1$, $\tfrac{1}{2}(M{+}1)(M{+}2)$, and $\tfrac{1}{6}(M{+}1)(M{+}2)(M{+}3)$ for $N = 1, 2, 3, 4$. Assuming that the formula is valid for M quanta and N oscillators, by the arguments above the allowable configurations for M quanta and $N + 1$ oscillators will be

$$\sum_{p=0}^{M}\binom{p+N-1}{p} = \binom{M+N}{M},$$

where the equality here follows from the *hockey-stick identity* from combinatorics.

Comments. The number M of quanta determines the total energy of the *system*, the system being our collection of N oscillators. The precise identification is a bit strange and relies on the quantum mechanics of a 1-dimensional harmonic oscillator. First, an oscillator has an *angular frequency* ω; think of ω as how fast it oscillates. We assume all of the oscillators in the collection have the same ω. The energy E_k of the kth oscillator is then

$$E_k = \hbar\omega(n_k + \tfrac{1}{2}),$$

where $\hbar$ is a fundamental constant of nature ($2\pi\hbar$ is Planck's *quantum of action*) and n_k is the number of quanta in the kth oscillator. Notice that when $n_k = 0$ (no quanta in the oscillator), the oscillator still has a nonzero energy $\tfrac{1}{2}\hbar\omega$. This is a peculiar aspect of quantum mechanics! The total energy E of the system is therefore

$$E = \sum_{k=1}^{N} E_k = \sum_{k=1}^{N}(n_k + \tfrac{1}{2})\hbar\omega,$$

In the "microcanonical ensemble" the total energy E is held fixed. We write this fixation as

$$\sum_{k=1}^{N} n_k = \frac{E}{\hbar\omega} - \frac{N}{2} \equiv M,$$

with M the total number of quanta in all oscillators. This means that $E = E_M$ depends on M and can only take discrete values. We want to count the number $\Omega(E, N)$ of allowable configurations for an attainable fixed energy $E = E_M$. In the parlance of quantum statistical mechanics $\Omega(E, N)$ is the *number of microstates*. We have solved this problem above with the "stars and bars" method:

$$\Omega(E, N) = \frac{(M+N-1)!}{M!(N-1)!} = \frac{(E/(\hbar\omega) + \tfrac{1}{2}N - 1)!}{(E/(\hbar\omega) - \tfrac{1}{2}N)!(N-1)!}$$

3. Simplify the expression $\sqrt{3+2\sqrt{2}} - \sqrt{3-2\sqrt{2}}$.

Preface. Before presenting solutions, we note that on non-negative real numbers the square-root function is increasing; that is, for $0 \le x_1 < x_2$ we have $0 \le \sqrt{x_1} < \sqrt{x_2}$. Put differently, the square-root function preserves order relations between non-negative real numbers. From this statement and the observation $8 < 9$, we conclude that $2\sqrt{2} < 3$. It follows that

$$0 < 3 - 2\sqrt{2} < 3 + 2\sqrt{2},$$

and, again using the statement about the square-root function, that

$$0 < \sqrt{3-2\sqrt{2}} < \sqrt{3+2\sqrt{2}}.$$

We conclude that *the expression in the problem statement is a strictly positive number.*

Solution 1. Squaring the expression gives

$$\begin{aligned}\left(\sqrt{3+2\sqrt{2}} - \sqrt{3-2\sqrt{2}}\right)^2 &= (3+2\sqrt{2}) - 2\sqrt{3+2\sqrt{2}}\sqrt{3-2\sqrt{2}} + (3-2\sqrt{2}) \\ &= 6 - 2\sqrt{(3+2\sqrt{2})(3-2\sqrt{2})} \\ &= 6 - 2\sqrt{9-8} \\ &= 4.\end{aligned}$$

This shows that the expression is either 2 or -2. However, we know from the argument above that the expression is positive, so the expression must equal 2.

Solution 2. Several students presented this alternative solution. Consider squaring the *positive* numbers $\sqrt{2}-1$ and $\sqrt{2}+1$. We find

$$(\sqrt{2}-1)^2 = 3 - 2\sqrt{2}, \qquad (\sqrt{2}+1)^2 = 3 + 2\sqrt{2},$$

showing that

$$\sqrt{3-2\sqrt{2}} = \sqrt{\left(\sqrt{2}-1\right)^2} = \sqrt{2}-1, \quad \sqrt{3+2\sqrt{2}} = \sqrt{\left(\sqrt{2}+1\right)^2} = \sqrt{2}+1.$$

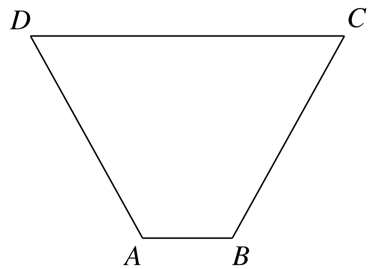
With these formulas.

$$\sqrt{3+2\sqrt{2}} - \sqrt{3-2\sqrt{2}} = \left(\sqrt{2}+1\right) - \left(\sqrt{2}-1\right) = 2.$$

Comments. For $x > 0$, we have $0 < y = \sqrt{x}$ if, and only if, $y^2 = x$. That $y^2 = x$ has a unique solution $y > 0$ for each real number $x > 0$ is a deep fact, rooted in core properties of the real numbers. For example, $\sqrt{1} = 1$ since $1^2 = 1$, $\sqrt{9} = 3$ since $3^2 = 9$, $\sqrt{\frac{1}{25}} = \frac{1}{5}$ since $(\frac{1}{5})^2 = \frac{1}{25}$. In the solutions we encountered a few NON-RULES for square roots, that were nonetheless proposed for this problem, leading to incorrect answers. For $x_1, x_2 > 0$:

- It is FALSE that $\sqrt{x_1+x_2} = \sqrt{x_1} + \sqrt{x_2}$. The example $x_1 = x_2 = 1$ illustrates this point, since $\sqrt{2} = 1.4\cdots \ne 2 = 1 + 1 = \sqrt{1} + \sqrt{1}$. However, the inequality $\sqrt{x_1+x_2} < \sqrt{x_1} + \sqrt{x_2}$ holds for all $x_1, x_2 > 0$ (why?).
- It is FALSE that if in addition $x_1 > x_2$, then $\sqrt{x_1+x_2} - \sqrt{x_1-x_2} = \sqrt{2x_2}$. The example $x_1 = 2$, $x_1 = 1$ illustrates this point, because $\sqrt{3} - \sqrt{1} = \sqrt{3} - 1 = 1.73\cdots - 1 = 0.73\cdots \ne 1.4\cdots = \sqrt{2}$. However, the inequality $\sqrt{x_1+x_2} - \sqrt{x_1-x_2} < \sqrt{2x_2}$ holds for all $x_1 > x_2 > 0$ (why?).

4. In the figure the isosceles trapezoid $ABCD$ has side AB parallel to side CD, sides AD and BC are of equal length, and the diagonals AC and BD are perpendicular. If the length of the side AB is 1 and the length of the side AD is 5, find the length of the side CD.



Solution 1. Let the segments AC and BD intersect at the point O. Then AOB is a $45° - 90° - 45°$ triangle, with base length 1. The sides AO and BO have length $\sqrt{\frac{1}{2}}$. Then AOD is a right triangle with hypotenuse AD of length 5 and side AO of length $\sqrt{\frac{1}{2}}$. The other side OD has length $\sqrt{25 - \frac{1}{2}} = 7\sqrt{\frac{1}{2}}$. Triangle DOC is then also a $45° - 90° - 45°$ triangle, with two sides of length $7\sqrt{\frac{1}{2}}$. The hypotenuse DC then has length $7\sqrt{\frac{1}{2}}\Big/\sqrt{\frac{1}{2}} = 7$.

Solution 2. Inspired by the work of Joshua Bala, 8th grade, Mandela International Magnet School. Refer to Fig. 2. Let $y = |CD|$, the length we are trying to compute. Let G be the point on CD such that BG is parallel to AD. Since AB is parallel to CD, it must be that $|BG| = |AD| = 5$ and that $|DG| = |AB| = 1$. Therefore, $|CG| = |CD| - 1 = y - 1$. Since the trapezoid is isosceles, $|BG| = |AD| = |BC|$, and so the triangle ΔGBC is isosceles. Let H be the foot of the perpendicular line through B to the segment CD (the black dotted line in the figure). Since ΔGBC is isosceles, H is halfway between G and C, so that $|HG| = |CH| = |CG|/2 = (y - 1)/2$. We can calculate $|BH|$ from the hypothesis that the diagonals are perpendicular, which we have not used yet. Indeed, let O be the intersection point of the diagonals, the right triangles ΔAOB and ΔCOD are similar isosceles triangles (inherited from the hypothesis that the trapezoid is isosceles), with each a $45° - 90° - 45°$ triangle. Whence their heights are half the length of their bases (respectively $|AB| = 1$ and $|CD| = y$). The sum of these heights is therefore $(1 + y)/2$. Note that these heights form a segment (the blue dotted line in the figure) joining the middle point of AB with the middle point of CD; this segment is parallel to BH and of equal length, hence $|BH| = (y + 1)/2$. By the Pythagorean theorem on the right triangle ΔCHB,

$$25 = |BC|^2 = |BH|^2 + |CH|^2 = \frac{(y+1)^2}{4} + \frac{(y-1)^2}{4} = \frac{y^2+1}{2},$$

whence $y^2 = 50 - 1 = 49$ and $y = 7$. Although we have found the answer, we may further compute $|BH| = (y + 1)/2 = 4$ and $|CH| = (y - 1)/2 = 3$. The triangle ΔBCH is a 5-3-4 triangle, as a number of students guessed, including Joshua, but this is a consequence of the hypothesis that the diagonals are perpendicular. If you guessed correctly, then you concluded that $|CH| = |HG| = 3$ and $y = |CH| + |HG| + |GD| = 3 + 3 + 1 = 7$, and got the right answer by chance!

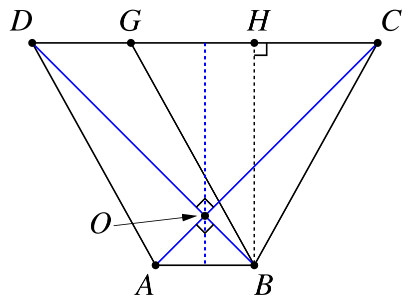


FIGURE 2.

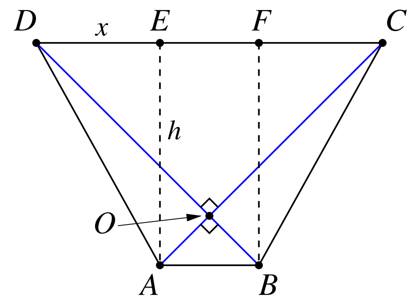


FIGURE 3.

Solution 3. Inspired by the work of RY PEPPER, 10th grade, Moreno Valley High School. Here we look at $\mathcal{A} = \mathcal{A}(ABCD)$, the area of the trapezoid, in two different ways. Refer to Fig. 3. First, let E and F be the feet of the perpendiculars dropped from A and B onto CD, respectively (note that F is the H from **Solution 2**, but $E \neq G$). Let $h = |AE| = |BF|$ and $x = |ED| = |CF|$. Then, because $|AB| = |FE| = 1$, we have $|CD| = |CF| + |FE| + |ED| = x + 1 + x = 1 + 2x$, and the area of the whole trapezoid is

$$\mathcal{A} = \frac{|AB| + |CD|}{2} h = \frac{1 + 1 + 2x}{2} h = (1 + x)h.$$

The triangle ΔDEA is a right triangle, whence by the Pythagorean theorem,

$$25 = |AD|^2 = |DE|^2 + |AE|^2 = x^2 + h^2,$$

and so $h = \sqrt{25 - x^2}$. With this result, the area is a function

$$\mathcal{A} = (1 + x)\sqrt{25 - x^2} \tag{1}$$

solely of x. Second, the area of the trapezoid is the sum of the areas of the triangles ΔADC and ΔABC. Let O be the intersection point of the diagonals, which are assumed to be perpendicular to each other. Then, as seen in Fig. 3, we have $\mathcal{A}(ADC) = \frac{1}{2}|AC| \cdot |DO|$ and $\mathcal{A}(ABC) = \frac{1}{2}|AC| \cdot |BO|$, so that

$$\mathcal{A} = \mathcal{A}(\Delta ADC) + \mathcal{A}(\Delta ABC) = |AC| \frac{|DO| + |OB|}{2} = \frac{|AC| \cdot |DB|}{2} = \frac{|AC|^2}{2}.$$

Now, the triangle ΔACE is a right isosceles triangle, because the angle $\angle EAC = 90° - \angle CAB = 45°$. Therefore, $|AC| = \sqrt{2}|EC| = \sqrt{2}(|EF| + |FC|) = \sqrt{2}(1 + x)$. We then have

$$\mathcal{A} = \frac{|AC|^2}{2} = \frac{2(1 + x)^2}{2} = (1 + x)^2. \tag{2}$$

Equating Eqs. (1) and (2), we conclude that $(1 + x)^2 = (1 + x)\sqrt{25 - x^2}$. Since $x > 0$, the factor $1 + x > 0$, and so it can be canceled to reach $(1 + x) = \sqrt{25 - x^2}$. Squaring both sides, we get $(1 + x)^2 = 25 - x^2$. Expanding the square on the left-hand side, we get $1 + 2x + x^2 = 25 - x^2$. Finally, subtraction of 25 and addition of x^2 on both sides yields $2x^2 + 2x - 24 = 0$. The left-hand side can be factored, giving $(x - 3)(2x + 8) = 0$, and the solutions to this equation are $x = 3$ or $x = -\frac{1}{4}$. Since $x > 0$, we must choose $x = 3$. We conclude that

$$h = \sqrt{25 - x^2} = \sqrt{25 - 9} = \sqrt{16} = 4, \quad \text{and} \quad |CD| = 1 + 2x = 1 + 6 = 7.$$

From Eqs. (1) and (2), we get $(1 + x)h = (1 + x)^2$, and deduce $h = x + 1 = 3 + 1 = 4$.

5. A sequence $a_1, a_2, a_3, \ldots$ of numbers is said to be an *arithmetic progression* if each term (other than the first) is the previous term plus a fixed number r. This means $a_2 = a_1 + r$, $a_3 = a_2 + r$, and, generally, $a_n = a_{n-1} + r$ for $n > 1$. For example, the sequence $3, 7, 11, 15, 19, 23, 27, 31, 35, \ldots$ is an arithmetic progression with $r = 4$ (presuming the pattern continues). The first three terms of an arithmetic progression of positive numbers are

$$a_1 = \tan x, \qquad a_2 = \cos x, \qquad a_3 = \sec x,$$

for some angle x in the first quadrant.

(a) What is the angle x?

(b) What is r?

(c) What position does $\cot x$ occupy in the sequence?

Solution 1. The assumed arithmetic progression gives the equations

$$\cos x = \tan x + r, \qquad \sec x = \cos x + r, \tag{3}$$

or, since $\tan x = \sin x / \cos x$ and $\sec x = 1/\cos x$,

$$\cos^2 x = \sin x + r\cos x, \qquad 1 = \cos^2 x + r\cos x.$$

Subtraction of the last two expressions yields

$$1 - \cos^2 x = \cos^2 x - \sin x$$

or $2\sin^2 x + \sin x - 1 = 0$ which has solutions $\sin x = -\frac{1}{4} \pm \frac{1}{4}\sqrt{9} = -1, \frac{1}{2}$, or $x = \frac{3}{2}\pi$, $\frac{1}{6}\pi$ (modulo 2π). Now go back to one of the above equations, to find $x = \frac{3}{2}\pi$ will not work. Indeed, both $\tan x$ and $\sec x$ are undefined at this value of x. In any case, $x = \frac{3}{2}\pi$ is not in the first quadrant, so our angle is $x = \frac{1}{6}\pi$. Now $\cos\frac{1}{6}\pi = \frac{1}{2}\sqrt{3}$, so from the last equation in (3)

$$\frac{2}{\sqrt{3}} = \frac{\sqrt{3}}{2} + r \implies r = \frac{2}{\sqrt{3}} - \frac{\sqrt{3}}{2} = \frac{4-3}{2\sqrt{3}} = \frac{1}{2\sqrt{3}} \implies r = \frac{\sqrt{3}}{6}.$$

For the final part, we check

$$a_4 = \sec(\tfrac{1}{6}\pi) + \frac{\sqrt{3}}{6} = \frac{2}{\sqrt{3}} + \frac{\sqrt{3}}{6} = \frac{4\sqrt{3}+\sqrt{3}}{6} = \frac{5\sqrt{3}}{6}.$$

But this is not

$$\cot x = \frac{\cos x}{\sin x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

Let's try then

$$a_5 = a_4 + r = \frac{5\sqrt{3}}{6} + \frac{\sqrt{3}}{6} = \sqrt{3}.$$

So $\cot x$ occupies the 5th position.

Solution 2. A number of students rightfully guessed that the angle they were looking for, must be one of the canonical angles in the first quadrant: $0°, 30°, 45°, 60°$, or $90°$. They experimented a little and concluded that $30°$ (which in radians is $\frac{1}{6}\pi$) is the correct angle. From then on they could figure out the rest of the problem. Note, however, that this approach leaves open the possibility that another angle in the first quadrant might also work. The first solution shows that $30°$ is the only first-quadrant angle yielding an arithmetic progression of the assumed form.

6. In how many ways can one choose a black square and a white square on an 8×8 chessboard, so that the chosen squares do not lie on the same row or on the same column?

Solution 1. We could notice that for each square Q there are $24 = 32 - 8$ squares of the opposite color that can be paired with Q. There are 64 squares Q, and every pair is counted twice, so the total number of pairs is $768 = \frac{1}{2}(24 \times 64)$. Many students avoided double counting by assuming that the "first" square of any pair is of a definite color (black or white, it does not matter). There are 32 squares of one color, and 24 of the other color that are not on the same row or column, for a total of $32 \times 24 = 768$ pairs.

If our chessboard had been 2024×2024, then we could still have used this method. For each square Q there are $\frac{1}{2}2024^2$ square of the opposite color. Of these 1012 occupy the same row as Q and 1012 occupy the same column. Whence there are

$$\frac{2024^2}{2} - 2024 = 2024(1012 - 1) = 2024 \times 1011 = 2,046,264$$

squares of the opposite color that can be paired with Q. There are 2024^2 squares Q and every pair is counted twice, so the total number of pairs is

$$\frac{2024^2 \times 2024 \times 1011}{2} = 2024^2 \times 1012 \times 1011.$$

Solution 2. Emily Yau, 9th grade, La Cueva High School, presented an argument using complementary counting. There are 32 black squares and 32 white squares, if we choose a black square first and a white square second, there are $32 \times 32 = 1024$ ways to choose the pairs (the choice of a definite color first avoids double counting). Let us find how many ways there are to pick squares so that they do lie on the same row or column. Each square has 8 squares of the opposite color that lie on the same row or column. Therefore, there are $32 \times 8 = 256$ ways to choose pairs of squares of opposite colors that lie on the same row/column. Subtracting, we find that there are then $1024 - 256 = 768$ ways to choose pairs of squares of opposite color that do not lie on the same row/column.

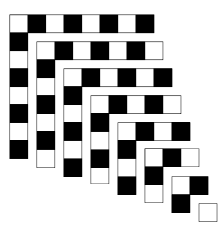
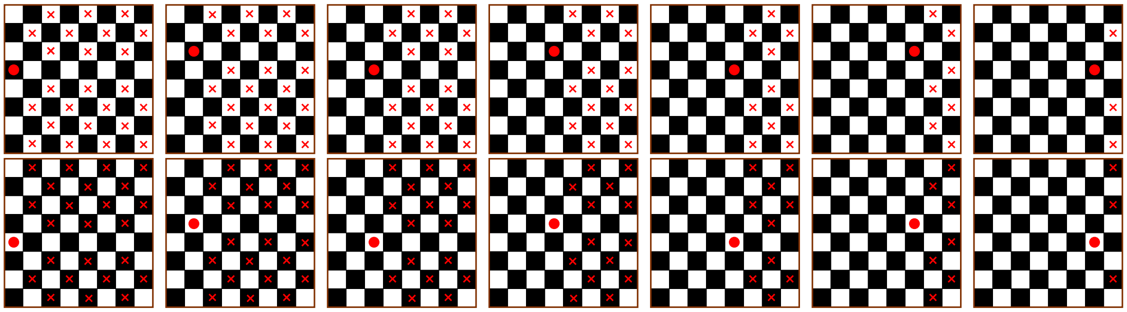


Figure 4. Refer to **Solutions 3 and 5**.

Solution 3. Adrian Medin, 8th grade, Los Alamos Middle School, partitioned the chessboard as shown in Fig. 4. Then we can count the number of white squares on consecutive decreasing inverted L's (that is, Γ's): 7, 7, 5, 5, 3, 3, 1, 1, and pair each with 24 allowed black squares. This process avoids double counting, and yields the total count

$$168 + 168 + 120 + 120 + 72 + 72 + 24 + 24 = 768.$$

Solution 4. Lydia Davis, 10th grade, Los Alamos High School, and Dylan Dencklau, 11th grade, El Dorado High School argued as follows.



(a) (b) (c) (d) (e) (f) (g)

FIGURE 5. Refer to **Solution 4**.

Our chessboard is 8×8 with 64 squares, 32 black and 32 white. We will count starting from the first column. Figure 5(a) depicts selection of a first square from the first column of the chessboard, with an example selection (the red dot) of both black and white shown. There are 8 possible selections, but whichever selection there are then 24 possible choices (the red crosses) for the second square of the opposite color which is neither in the first column nor in the row of the selected first square. Thus, our choices here allow for $192 = 8 \times 24$ pairs.

Likewise, Figures 5(b)-(g) depict selection of a first square from the second, third, fourth, fifth, sixth and seventh columns of the chessboard, with an example selection (the red dot) of both black and white shown. There are again 8 possible selections, but whichever selection there are then 21, 17, 14, 10, 7, and 3 possible choices (the red crosses) for the second square of the opposite color which is not in the second, third, fourth, fifth, sixth, and seventh columns, respectively, and also not in the row of the selected first square. By not allowing the second square to be drawn from the appropriate previous columns, we rule out pairs of squares already accounted for in the counting. Thus, our choices here allow for $168 = 8 \times 21$, $136 = 8 \times 17$, $112 = 8 \times 14$, $80 = 8 \times 10$, $56 = 8 \times 7$, and $24 = 8 \times 3$ pairs.

We have a total of $768 = 192 + 168 + 136 + 112 + 80 + 56 + 24$ pairs of squares, one white and one black, such that they are not on the same row or column. YUIKO YAMAGUSHI, 8th grade, Los Alamos Middle School, used this idea, working with rows instead of columns, and drew a picture worth a thousand words similar to this one:

24	24	24	24	24	24	24	24	8×24
21	21	21	21	21	21	21	21	8×21
17	17	17	17	17	17	17	17	8×17
14	14	14	14	14	14	14	14	8×14
10	10	10	10	10	10	10	10	8×10
7	7	7	7	7	7	7	7	8×7
3	3	3	3	3	3	3	3	8×3
0	0	0	0	0	0	0	0	8×0

Yuiko's picture had shading to distinguish black and white squares, and to facilitate counting!

Comment. Were our chessboard 2024×2024, this is not a method we would like to use to count the number of pairs. However, there is some rhyme to what has been done, and we might seek formulas in terms of n, the number of columns.

Solution 5. Aditi Ganti, 10th grade, La Cueva HS, gained insight by working with smaller-sized chessboards, seeking a pattern. First count the number L_n of pairs of black and white squares with at least one square in the first row or the first column (an inverted L, or Γ, shape; see Fig. 4) of an $n \times n$ chessboard, but with the squares not on the same row or column. To these, add the number of allowed pairs of black and white squares that can be drawn from the remaining $(n-1) \times (n-1)$ chessboard in the SW corner. Let p_n be the number of allowable pairs in an $n \times n$ chessboard. The following recursion formula holds:

$$p_n = L_n + p_{n-1}. \tag{4}$$

Let us see how this works for $n = 2, 3, \ldots, 8$, which some students attempted without formalizing the recurrence. When calculating L_n, we must avoid double counting pairs for which one square lies on the first row and the other on the first column.

2×2 chessboard: There are 0 such pairs. Hence $p_2 = 0$.

3×3 chessboard: Each square on the first column and the first row (5 of them) has exactly 2 squares of the opposite color left to pair with. We have double counted 2 pairs that have both squares in the first row and column, so $L_3 = 5 \times 2 - 2 = 10 - 2 = 8$. Afterwards, we have to search on a 2×2 chessboard, but $p_2 = 0$. Hence $p_3 = L_3 + p_2 = 8 + 0 = 8$.

4×4 chessboard: Each square on the first column and the first row (7 of them) has exactly 4 squares of the opposite color left to pair with. We have double counted 4 pairs that have both squares in the first row and column, so $L_4 = 7 \times 4 - 4 = 28 - 4 = 24$. Afterwards, we have to search on a 3×3 chessboard, but we know there are $p_3 = 8$ pairs left. Hence $p_4 = L_4 + p_3 = 24 + 8 = 32$.

5×5 chessboard: Each square on the first column and the first row (9 of them) has exactly 8 squares of the opposite color left to pair with. We have double counted 8 pairs that have both squares in the first row and column, so $L_5 = 8 \times 9 - 8 = 72 - 8 = 64$. Afterwards, we have to search on a 4×4 chessboard, and we know there are $p_4 = 32$ pairs left. Hence $p_5 = L_5 + p_4 = 64 + 32 = 96$.

6×6 chessboard: Each square on the first column and the first row (11 of them) has exactly 12 squares of the opposite color left to pair with. We have double counted 12 pairs that have both squares in the first row and column, so $L_6 = 11 \times 12 - 12 = 120$. Afterwards, we have to search on a 5×5 chessboard, and we know there are $p_5 = 96$ pairs left. Hence $p_6 = L_6 + p_5 = 120 + 96 = 216$.

7×7 chessboard: Each square on the first column and the first row (13 of them) has exactly 18 squares of the opposite color left to pair with. We have double counted 18 pairs that have both squares in the first row and column, so $L_7 = 13 \times 18 - 18 = 234 - 18 = 216$. Afterwards, we have to search on a 6×6 chessboard, and we know there are $p_6 = 216$ pairs left. Hence $p_7 = L_7 + p_6 = 216 + 216 = 432$.

8×8 chessboard: Each square on the first column and the first row (15 of them) has exactly 24 squares of the opposite color left to pair with. We have double counted 24 pairs that have both squares in the first row and column, so $L_8 = 15 \times 24 - 24 = 360 - 22 = 336$. Afterwards, we have to search on a 7×7 chessboard, and we know there are $p_7 = 432$ pairs left. Hence $p_8 = L_8 + p_7 = 336 + 432 = 768$.

For a large chessboard this step-by-step approach would prove too burdensome, so let us formalize what we have done explicitly for the cases $n = 2, 3, \dots, 8$. The number of squares in the first row and first column of an $n \times n$ chess board is $2n - 1$ (where the -1 avoids double counting of the NW corner square which lies on both the first row and first column). Let us refer to this set of squares as the *first-L* (although *first-Γ* would conjure up the better picture). After removing the row and column that intersect at a square Q in the first-L, we wish to count the number of opposite-color squares left over to be paired with Q.

When n is odd, there are $\frac{1}{2}(n-1)^2$ opposite-color squares left over to pair with Q, so the total number of squares that can be paired with squares in the first-L is naively $\frac{1}{2}(2n-1)(n-1)^2$. However, there are allowable pairs with *both* squares on the first-L, and these have been double counted. To correct, we must remove a factor of $\frac{1}{2}(n-1)^2$, yielding

$$L_n = \tfrac{1}{2}(2n-1)(n-1)^2 - \tfrac{1}{2}(n-1)^2 = (n-1)^3, \text{ provided } n \text{ is odd.}$$

This formula gives our earlier counts: $L_3 = 8 = 2^3$, $L_5 = 64 = 4^3$, and $L_7 = 216 = 6^3$. When n is even, there are $\frac{1}{2}[(n-1)^2 - 1]$ opposite-color squares that can be paired with a Q in the first-L, so the total number of squares that can be paired with squares in the first-L is naively $\frac{1}{2}(2n-1)[(n-1)^2-1]$. However, we have again double counted pairs for which both squares lie on the first-L. With the appropriate correction,

$$L_n = \tfrac{1}{2}(2n-1)[(n-1)^2-1] - \tfrac{1}{2}[(n-1)^2-1] = (n-1)^3 - (n-1), \text{ provided } n \text{ is even.}$$

We again get our earlier counts: $L_4 = 24 = 3^3 - 3$, $L_6 = 120 = 5^3 - 5$, and $L_8 = 336 = 7^3 - 7$.

We can now use the recurrence (4) and the fact that $p_2 = 0$ to conclude that

$$\begin{aligned} p_n &= L_3 + L_4 + L_5 + \cdots + L_n \\ &= \big(2^3 + 3^3 + 4^3 + \cdots + (n-1)^3\big) - \big(3 + 5 + 7 + \cdots + \lfloor n/2 \rfloor\big) \\ &= \big(1^3 + 2^3 + 3^3 + 4^3 + \cdots + (n-1)^3\big) - \big(1 + 3 + 5 + 7 + \cdots + (2k-1)\big). \end{aligned}$$

Here k depends on the parity of n: if n is even, then $n = 2k$, and if n is odd, then $n = 2k-1$. With floor notation, $k = \lfloor n/2 \rfloor$, where $\lfloor x \rfloor$ is the largest integer smaller or equal to x. The sum of the first N cubes can be checked by induction to be the square of the sum of the first N natural numbers, $1^3 + 2^3 + \cdots + N^3 = (1 + 2 + 3 + \cdots + N)^2$. The sum of the first N natural numbers is $\frac{1}{2}N(N+1)$, that is $1 + 2 + 3 + \cdots + N = \frac{1}{2}N(N+1)$, as can be checked by a telescoping-sum argument, induction, or Gauss' summation trick. The sum of the first k odd integers is k^2, namely $1 + 3 + 5 + \cdots + (2k-1) = k^2$. This result can also be confirmed with telescoping sums or induction. Setting $N = n-1$ and $k = \lfloor n/2 \rfloor$, we conclude that

$$p_n = \tfrac{1}{4}(n-1)^2 n^2 - \lfloor \tfrac{1}{2} n \rfloor^2 = \begin{cases} \tfrac{1}{4}(n-2)n^3 & \text{if } n \text{ is even} \\ \tfrac{1}{4}(n+1)(n-1)^3 & \text{if } n \text{ is odd.} \end{cases}.$$

As a sanity check, let us verify that this formula yields the correct p_8:

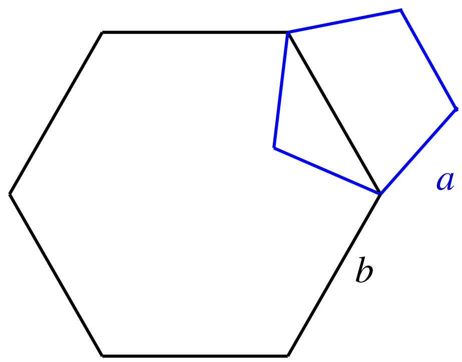
$$p_8 = \tfrac{1}{4}(8-1)^2 \cdot 8^2 - \lfloor \tfrac{1}{2} \cdot 8 \rfloor^2 = \tfrac{1}{4} 7^2 \cdot 8^2 - 4^2 = 49 \cdot 16 - 16 = 784 - 16 = 768.$$

You may verify that it also gives correct p_3 through p_7 found above, as well as the correct p_{2024} from **Solution 1**.

7. In the figure both the hexagon and pentagon are regular; each has sides which are equal.

(a) Show that $\cos(\frac{1}{5}\pi) = \frac{1}{4}(\sqrt{5}+1)$ and $\cos(\frac{2}{5}\pi) = \frac{1}{4}(\sqrt{5}-1)$.

(b) Using the results from **(a)**, find the ratio of the area of the large hexagon (with side length b) and the area of the small pentagon (with side length a).



Preface. The angles encountered in this problem are $\frac{1}{10}\pi = 18°$, $\frac{1}{5}\pi = 36°$, $\frac{3}{10}\pi = 54°$, $\frac{2}{5}\pi = 72°$, $\frac{3}{5}\pi = 108°$, and $\frac{4}{5}\pi = 144°$. We choose to work exclusively with radians. Notice that $0 < \frac{1}{10}\pi < \frac{1}{5}\pi < \frac{3}{10}\pi < \frac{2}{5}\pi < \frac{1}{2}\pi$. Therefore, the angles $\frac{1}{10}\pi$, $\frac{1}{5}\pi$, $\frac{3}{10}\pi$, and $\frac{2}{5}\pi$ all correspond to the first quadrant; the sine or cosine of any of these angles is strictly positive.

First solution to (a). Let $x = \cos(\frac{1}{5}\pi)$ and $y = \cos(\frac{2}{5}\pi)$. Notice that $\frac{4}{5}\pi = \pi - \frac{1}{5}\pi$, or $\frac{4}{5}\pi - \frac{1}{2}\pi = \frac{1}{2}\pi - \frac{1}{5}\pi$. Since cosine is an odd function relative to $\frac{1}{2}\pi$, we see that $\boxed{\cos(\frac{4}{5}\pi) = -x}$. Figure 6 demonstrates this identity graphically. Another way to get this result relies on the

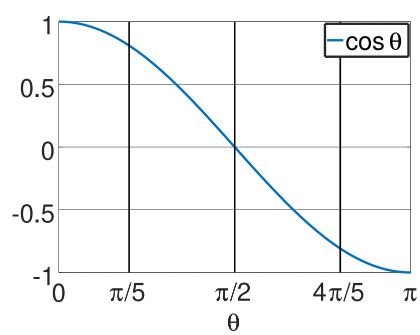


FIGURE 6. Function $\cos\theta$ near $\theta = \pi/2$.

addition-of-angle formula $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ to write

$$\cos(\pi - \tfrac{1}{5}\pi) = \cos\pi\cos\tfrac{1}{5}\pi,$$

again giving the boxed result.

Now use the double-angle formula $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ to write

$$x^2 = \tfrac{1}{2}(1+\cos(\tfrac{2}{5}\pi)) = \tfrac{1}{2}(1+y), \qquad y^2 = \tfrac{1}{2}(1+\cos(\tfrac{4}{5}\pi)) = \tfrac{1}{2}(1-x). \tag{5}$$

Subtraction of the second formula from the first gives

$$(x+y)(x-y) = \tfrac{1}{2}(x+y).$$

Since both $x > 0$ and $y > 0$, we have $x + y > 0$, and the last equation implies $x - y = \frac{1}{2}$, that is $y = x - \frac{1}{2}$. Substitution of this result into the first equation from (5) gives

$$x^2 = \tfrac{1}{2}(1+y) = \tfrac{1}{2}(\tfrac{1}{2}+x).$$

Rearrangement then yields the quadratic equation

$$4x^2 - 2x - 1 = 0.$$

The roots of this equation are $\frac{1}{4} \pm \frac{1}{4}\sqrt{5}$, and $x = \frac{1}{4}(\sqrt{5}+1)$ necessarily, since $x > 0$. Then $y = x - \frac{1}{2} = \frac{1}{4}(\sqrt{5}-1)$.

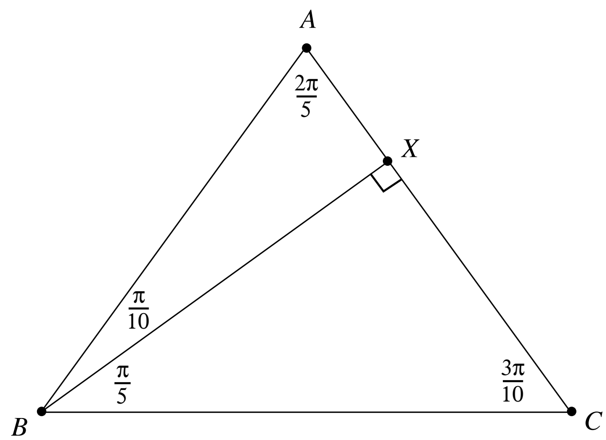


FIGURE 7. $\frac{3}{10}\pi$–$\frac{2}{5}\pi$–$\frac{3}{10}\pi$ isosceles triangle.

Second solution to (a), due to AKILAN SANKARAN, 12th grade, Albuquerque Academy. Consider the $\frac{3}{10}\pi$–$\frac{2}{5}\pi$–$\frac{3}{10}\pi$ isosceles triangle shown in Fig. 7, assuming that the length $|BC|$ of the segment BC is 1. Then from the figure $|BX| = \cos(\frac{1}{5}\pi)$ and $|CX| = \sin(\frac{1}{5}\pi)$. Also from the figure, $|BX| = |AB|\cos(\frac{1}{10}\pi)$, showing that

$$|AB| = \frac{|BX|}{\cos(\frac{1}{10}\pi)} = \frac{\cos(\frac{1}{5}\pi)}{\cos(\frac{1}{10}\pi)}.$$

It then follows that

$$|AX| = |AB|\sin(\tfrac{1}{10}\pi) = \cos(\tfrac{1}{5}\pi)\tan(\tfrac{1}{10}\pi).$$

Since ΔBAC is isosceles, $|AB| = |AC|$, implying that $|AB| = |AX| + |CX|$. With the trigonometric expressions found above, we write this identity as

$$\frac{\cos(\frac{1}{5}\pi)}{\cos(\frac{1}{10}\pi)} = \cos(\tfrac{1}{5}\pi)\tan(\tfrac{1}{10}\pi) + \sin(\tfrac{1}{5}\pi),$$

or, with the definition of tangent,

$$\cos(\tfrac{1}{5}\pi)\Big[\frac{\sin(\frac{1}{10}\pi)}{\cos(\frac{1}{10}\pi)} - \frac{1}{\cos(\frac{1}{10}\pi)}\Big] + \sin(\tfrac{1}{5}\pi) = 0.$$

The value of sine or cosine on any of the angles here lies strictly between 0 and 1; see the preface. Let $x = \cos(\frac{1}{5}\pi)$, so that $\cos(\frac{1}{10}\pi) = \sqrt{\frac{1}{2}(x+1)}$ and $\sin(\frac{1}{10}\pi) = \sqrt{\frac{1}{2}(1-x)}$. With these expressions, the last equation becomes

$$x\left[\sqrt{\frac{1-x}{1+x}} - \sqrt{\frac{2}{1+x}}\right] + \sqrt{1-x^2} = 0,$$

and, upon rearrangement,

$$\sqrt{1-x^2}\left(\frac{1+2x}{1+x}\right) = x\sqrt{\frac{2}{1+x}}.$$

Overall multiplication by $\sqrt{x+1}$ yields

$$\sqrt{1-x}(2x+1) = x\sqrt{2}.$$

We then find

$$\sqrt{\frac{1-x}{2}} = \frac{x}{2x+1}.$$

Upon squaring this equation and subsequent simplification, we reach

$$(x+1)(4x^2-2x-1) = 0.$$

The roots here are -1, $\frac{1}{4}(1-\sqrt{5})$, and $\frac{1}{4}(1+\sqrt{5})$. Since $0 < x < 1$, we conclude that

$$x = \tfrac{1}{4}(1+\sqrt{5}).$$

Finally, as in the first solution to part **(a)**, we use a double-angle formula to compute

$$\cos(\tfrac{2}{5}\pi) = 2x^2 - 1 = \tfrac{1}{4}(\sqrt{5}-1).$$

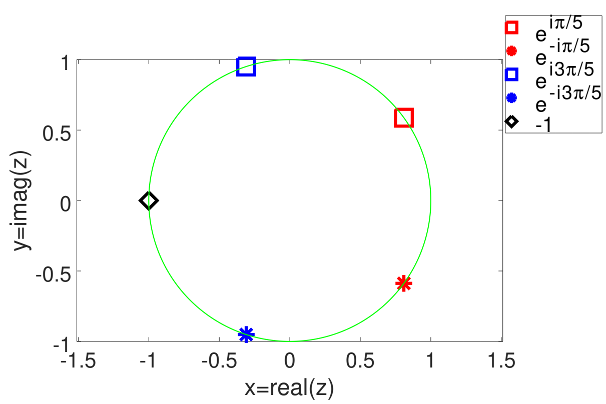


Figure 8. Roots of $z^5 + 1 = 0$ on the unit circle. The roots define a pentagon!

Third solution to (a), due to Hiro Jau, 11th grade, La Cueva High School. This approach uses complex numbers. Consider the equation $z^5+1 = 0$, for which the first-quadrant number $z = e^{i\pi/5}$ is an obvious solution. None of the other solutions (namely -1, $e^{-i\pi/5}$, $e^{-i3\pi/5}$, and $e^{i3\pi/5}$) lie in the first quadrant. Each of these solutions is a unit-modulus complex number, and all are shown in Fig. 8.

Claim: The quintic polynomial z^5+1 can be factored as

$$z^5+1=\overbrace{(z+1)}^{\text{root }-1}\overbrace{\Big(z^2-\frac{1+\sqrt{5}}{2}z+1\Big)}^{\text{roots in quadrants 1,2}}\overbrace{\Big(z^2-\frac{1-\sqrt{5}}{2}z+1\Big)}^{\text{roots in quadrants 3,4}}.$$

Proof. We begin with

$$z^5+1=(z+1)(z^4-z^3+z^2-z+1),$$

and then observe that

$$\frac{1}{z^2}(z^4-z^3+z^2-z+1)=z^2-z+1-\frac{1}{z}+\frac{1}{z^2}=\Big(z+\frac{1}{z}\Big)^2-\Big(z+\frac{1}{z}\Big)-1.$$

The roots of u^2-u-1 are $u_\pm=\frac{1}{2}(1\pm\sqrt{5})$, and it then follows that

$$z^4-z^3+z^2-z+1=\Big(z^2-\frac{1+\sqrt{5}}{2}z+1\Big)\Big(z^2-\frac{1-\sqrt{5}}{2}z+1\Big).$$

This identity may also be confirmed by expansion of the right-hand side. Finally, let us confirm the overbrace statements about the location of the roots. That $z+1$ has $z=-1$ as its root is clear. For the middle factor $z^2-\frac{1}{2}(1+\sqrt{5})z+1$, the quadratic formula gives roots

$$\frac{1+\sqrt{5}}{4}\pm\frac{1}{2}\sqrt{\Big(\frac{1+\sqrt{5}}{2}\Big)^2-4}=\frac{1+\sqrt{5}}{4}\pm\frac{\mathrm{i}}{4}\sqrt{10-2\sqrt{5}}.$$

Since they have strictly positive real part, these roots lie in the first and fourth quadrant respectively; in fact, these are $e^{\mathrm{i}\pi/5}$ and $e^{-\mathrm{i}\pi/5}$. Similar calculation shows that the roots of the final factor lie in the second and third quadrants; these are $e^{3\mathrm{i}\pi/5}$ and $e^{-3\mathrm{i}\pi/5}$. □

With the claim, we conclude that

$$e^{\mathrm{i}\pi/5}=\frac{1+\sqrt{5}}{4}+\frac{\mathrm{i}}{4}\sqrt{10-2\sqrt{5}}.$$

Finally, with the Euler formula $e^{\mathrm{i}\pi/5}=\cos(\frac{1}{5}\pi)+\mathrm{i}\sin(\frac{1}{5}\pi)$, we find, upon taking the real part of both sides, that

$$\cos(\tfrac{1}{5}\pi)=\tfrac{1}{4}(1+\sqrt{5}).$$

As in the earlier solutions, with this result use of a double-angle formula yields the stated expression for $\cos(\frac{2}{5}\pi)$.

Fourth Solution to (a), due to Grace Hsieh, 10th grade, La Cueva High School. This approach uses the Law of Cosines. On the isosceles triangle with sides b,a,b and opposite angles $\frac{2}{5}\pi,\frac{1}{5}\pi,\frac{2}{5}\pi$, we have, by the Law of Cosines, that

$$\begin{aligned}a^2&=2b^2-2b^2\cos(\tfrac{1}{5}\pi)\\ b^2&=a^2+b^2-2ab\cos(\tfrac{2}{5}\pi).\end{aligned}$$

From the second equation we get $a=2b\cos(\frac{2}{5}\pi)$, which we can substitute into the first equation, thereby finding

$$4b^2\cos^2(\tfrac{2}{5}\pi)=2b^2\big(1-\cos(\tfrac{1}{5}\pi)\big).$$

Dividing by $2b^2 > 0$ and using the double-angle formula $\cos(\frac{2}{6}\pi) = 2\cos^2(\frac{1}{5}\pi) - 1$, we get the following equation involving solely $\cos(\frac{1}{5}\pi)$:

$$2\big[2\cos^2(\tfrac{1}{5}\pi) - 1\big]^2 = 1 - \cos(\tfrac{1}{5}\pi).$$

With $x = \cos(\frac{1}{5}\pi)$, the equation becomes $2(2x^2-1)^2 = 1 - x$, and, after some algebra, $8x^4 - 8x^2 + x + 1 = 0$. Clearly, $x = -1$ is a solution of this equation (as $8 - 8 - 1 + 1 = 0$), and so is $x = \frac{1}{2}$ (as $\frac{8}{16} - \frac{8}{4} + \frac{1}{2} + 1 = 0$). We conclude that

$$8x^4 - 8x^2 + x + 1 = (x+1)(2x-1)(4x^2 - 2x - 1).$$

Grace factored the equation as $8x^4 - 8x^2 + x + 1 = (8x^3 - 8x^2 + 1)(x+1)$, and then verified that $x = \frac{1}{4}(1+\sqrt{5})$ is a solution. We can use the quadratic formula to identify $x = \frac{1}{4}(1 \pm \sqrt{5})$ as two roots, along with $x = -1, \frac{1}{2}$. We aim to identify one of the roots as $\cos(\frac{1}{5}\pi)$, which is strictly positive. The negative roots are summarily discarded, and we are left to consider $x = \frac{1}{4}(1+\sqrt{5})$ and $x = \frac{1}{2}$. The identification "$\cos(\frac{1}{5}\pi) = \frac{1}{2}$" is not possible. Why? We know that $0 < \frac{1}{5}\pi < \frac{1}{3}\pi < \frac{1}{2}\pi$, and the cosine is a decreasing function on the interval $(0, \frac{1}{2}\pi)$; whence $\cos(\frac{1}{5}\pi) > \cos(\frac{1}{3}\pi) = \frac{1}{2}$. It must be that $\cos(\frac{1}{5}\pi) = \frac{1}{4}(1+\sqrt{5})$. As before, with this result use of a double-angle formula yields the stated expression for $\cos(\frac{2}{5}\pi)$.

Solution to (b). The intersection between the hexagon and the pentagon is a $\frac{1}{5}\pi - \frac{3}{5}\pi - \frac{1}{5}\pi$ triangle with side lengths $a - b - a$. So $\frac{1}{2}b = a\cos(\frac{1}{5}\pi)$, and from part **(a)**

$$b^2 = 4a^2\cos^2(\tfrac{1}{5}\pi) = \tfrac{1}{2}a^2(\sqrt{5}+3).$$

The area of the pentagon is $A_P = \alpha_P a^2$, and the area of the hexagon is $A_H = \alpha_H b^2$, where, respectively, α_P and α_H are the areas of a pentagon of unit side length and a hexagon of unit side length. With the last equation, the ratio in question is the following.

$$\boxed{\frac{A_H}{A_P} = \frac{\alpha_H}{\alpha_P}\frac{b^2}{a^2} = \frac{\alpha_H}{\alpha_P}\frac{\sqrt{5}+3}{2}}$$

To complete the solution, we need to work out both α_H and α_P. The factor α_H is the area of six equilateral triangles of unit side length. Thus, α_H is the area of twelve right triangles with base lengths $\frac{1}{2}\sqrt{3}$, $\frac{1}{2}$ and hypotenuse length 1. Each of these twelve triangles has area $\frac{1}{8}\sqrt{3}$, and $\boxed{\alpha_H = \frac{1}{2}3\sqrt{3}}$. The factor α_P is more difficult to work out, so let us state the result.

Claim: $\boxed{\alpha_P = \frac{1}{4}\sqrt{5(5+2\sqrt{5})}}$.

Proof. We give three proofs of the claim, one here and two more at the end in an appendix. The factor α_P is area of 5 isosceles triangles with base length 1 and angles $54° - 72° - 54°$, namely $\frac{3}{10}\pi - \frac{2}{5}\pi - \frac{3}{10}\pi$. Thus, α_P is the area of 10 right triangles with angles $\frac{2}{5}\pi - \frac{1}{2}\pi - \frac{3}{10}\pi$, with $\frac{1}{2}$ the base length opposite the angle $\frac{2}{5}\pi$; see Fig. 9. If the other base length is h, known as the *apothem*, then the hypotenuse has length $h/\cos(\frac{1}{2}\pi) = 4h/(\sqrt{5}+1)$. By the Pythagorean Theorem

$$\frac{1}{4} + h^2 = \frac{16h^2}{6+2\sqrt{5}} \iff \frac{1}{4} = (5-2\sqrt{5})h^2.$$

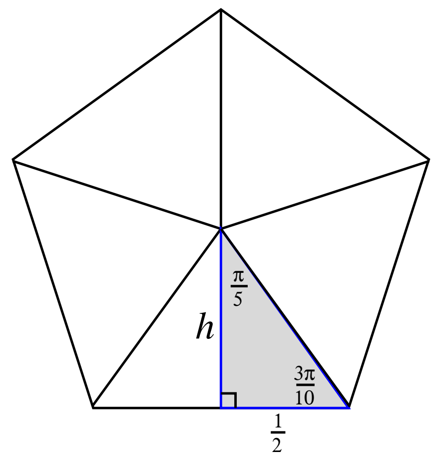


FIGURE 9. Area α_P as the area of 10 right triangles

Therefore, the other base length (the apothem) is

$$h = \frac{1}{2\sqrt{5-2\sqrt{5}}} = \frac{\sqrt{5+2\sqrt{5}}}{2\sqrt{5}} = \frac{\sqrt{5(5+2\sqrt{5})}}{10}.$$

We then have

$$\alpha_P = 10 \cdot \Big(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{5(5+2\sqrt{5})}}{10}\Big).$$

This expression reduces to the boxed one stated in the claim. □

Putting together the three boxed expressions above, we have the following.

$$\frac{A_H}{A_P} = 3(\sqrt{5}+3)\sqrt{\frac{3}{5(5+2\sqrt{5})}} = 3\sqrt{\frac{3(\sqrt{5}+3)^2}{5(5+2\sqrt{5})}} = 3\sqrt{\frac{3(14+6\sqrt{5})}{5(5+2\sqrt{5})}}$$
$$= 3\sqrt{\frac{3(14+6\sqrt{5})(5-2\sqrt{5})}{25}} = \tfrac{3}{5}\sqrt{6(5+\sqrt{5})}$$

The last expression is arguably the most simplified.

Appendix: Here we present two alternative computations of α_P. First consider Fig. 10 which depicts the area α_P as the area of three isosceles triangles. From the figure and part **(a)**, we see that $p = 2\cos(\frac{1}{5}\pi) = \frac{1}{2}(1+\sqrt{5})$. The area of each of the outer two triangles with side lengths $1-p-1$ is

$$A_{1p1} = 2\big[\tfrac{1}{2}\cos(\tfrac{1}{5}\pi)\sin(\tfrac{1}{5}\pi)\big] = \cos(\tfrac{1}{5}\pi)\sin(\tfrac{1}{5}\pi) = \tfrac{1}{16}(1+\sqrt{5})\sqrt{10-2\sqrt{5}},$$

where we have used the formula for $\cos(\frac{1}{5}\pi)$ and $\sin(\frac{1}{5}\pi) = \sqrt{1-\cos^2(\frac{1}{5}\pi)}$. Via a similar calculation, the area of the middle triangle with side lengths $p-1-p$ is

$$A_{p1p} = 2\big[\tfrac{1}{4}\cdot p\sin(\tfrac{2}{5}\pi)\big] = \cos(\tfrac{1}{5}\pi)\sin(\tfrac{2}{5}\pi) = \tfrac{1}{16}(1+\sqrt{5})\sqrt{10+2\sqrt{5}}.$$

With these results, we perform the following long, but straightforward, calculation which yields result stated in the claim.

$$
\begin{aligned}
\alpha_P &= 2A_{1p1} + A_{p1p} \\
&= \tfrac{1}{16}(1+\sqrt{5})\Big[2\sqrt{10-2\sqrt{5}} + \sqrt{10+2\sqrt{5}}\Big] \\
&= \tfrac{1}{16}(1+\sqrt{5})\Big[2 + \frac{10+2\sqrt{5}}{\sqrt{80}}\Big]\sqrt{10-2\sqrt{5}} \\
&= \tfrac{1}{64}(1+\sqrt{5})\Big(\frac{10+10\sqrt{5}}{\sqrt{5}}\Big)\sqrt{10-2\sqrt{5}} \\
&= \tfrac{1}{32}\sqrt{5}\big(1+\sqrt{5}\big)^2\sqrt{10-2\sqrt{5}} \\
&= \tfrac{1}{32}\sqrt{5}\sqrt{(56+24\sqrt{5})(10-2\sqrt{5})} \\
&= \tfrac{1}{32}\sqrt{5}\sqrt{320+128\sqrt{5}} \\
&= \tfrac{1}{4}\sqrt{5}\sqrt{5+2\sqrt{5}}
\end{aligned}
$$

For the second alternative calculation, view $\alpha_P = A_{1p1} + A_{\text{trapezoid}}$ as shown in Fig. 11. We

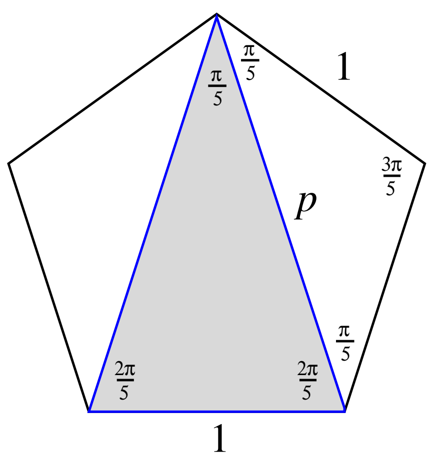


FIGURE 10. Area α_P as the area of three isosceles triangles.

have computed the area A_{1p1} above. The trapezoid has area $A_{\text{trapezoid}} = \frac{1}{2}(p+1)q$, with p as above and

$$q = \sin(\tfrac{2}{5}\pi) = \sqrt{1 - \tfrac{1}{16}(\sqrt{5}-1)^2} = \tfrac{1}{4}\sqrt{10+2\sqrt{5}}.$$

We than have

$$A_{\text{trapezoid}} = \tfrac{1}{16}(3+\sqrt{5})\sqrt{10+2\sqrt{5}},$$

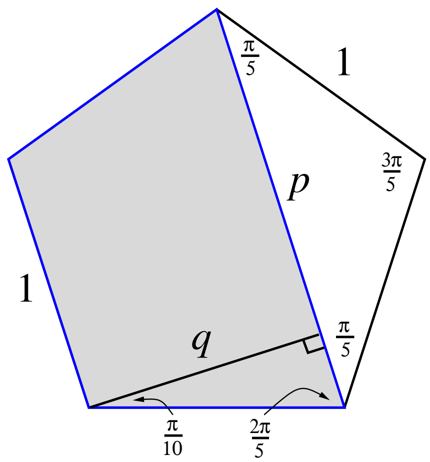


FIGURE 11. Area α_P as the area of an isosceles triangle and area of trapezoid.

from which

$$\begin{aligned}\alpha_P &= \tfrac{1}{16}(1+\sqrt{5})\sqrt{10-2\sqrt{5}} + \tfrac{1}{16}(3+\sqrt{5})\sqrt{10+2\sqrt{5}} \\ &= \tfrac{1}{16}(1+\sqrt{5})\left[2\sqrt{10-2\sqrt{5}} + \sqrt{10+2\sqrt{5}}\right] \\ &- \tfrac{1}{16}(1+\sqrt{5})\sqrt{10-2\sqrt{5}} + \tfrac{1}{8}\sqrt{10+2\sqrt{5}}.\end{aligned}$$

As shown in the first alternative calculation, the first term in the last expression is the result from the claim. Therefore, we have found that

$$\alpha_P = \tfrac{1}{4}\sqrt{5}\sqrt{5+2\sqrt{5}} - \tfrac{1}{16}\sqrt{(6+2\sqrt{5})(10-2\sqrt{5})} + \tfrac{1}{16}\sqrt{40+8\sqrt{5}}.$$

Now, $(6+2\sqrt{5})(10-2\sqrt{5}) = 40+8\sqrt{2}$, and the last two factors cancel. We have again established the claim.

8. **(a)** If the expression

$$(((x-2)^2-2)^2-2)^2,$$

with three pairs of parentheses, is multiplied out, what is the coefficient of x^2?

(b) If the expression

$$(\cdots(((x-2)^2-2)^2-2)^2-\cdots-2)^2,$$

with 2025 pairs of parentheses, is multiplied out, what is the coefficient of x^2?

Solution. For **(a)** a brute-force approach is possible. Denote the expression by p_3. Expansion gives

$$p_3 = (((x-2)^2-2)^2-2)^2 = \underbrace{((x-2)^2-2)^4}_{\text{term 1}} \underbrace{-4((x-2)^2-2)^2}_{\text{term 2}} +4$$

Using Pascal's triangle

$$\text{term 1} = (x-2)^8 + 4(x-2)^6(-2) + 6(x-2)^4(-2)^2 + 4(x-2)^2(-2)^3 + (-2)^4$$
$$\text{term 2} = -4\big[(x-2)^4 - 4(x-2)^2 + 4\big]$$

Collecting the results, we find

$$p_3 = (x-2)^8 - 8(x-2)^6 + 20(x-2)^4 - 16(x-2)^2 + 4.$$

Now, again using Pascal's triangle the coefficient of x^2 in $(x-2)^k$ is $\binom{k}{k-2}(-2)^{k-2} = \frac{1}{2}k(k-2)(-2)^{k-2}$. So we have the table

$(x-2)^8$: coefficient of x^2 is $\frac{1}{2}8\cdot 7\cdot(-2)^6 = 1792$.
$(x-2)^6$: coefficient of x^2 is $\frac{1}{2}6\cdot 5\cdot(-2)^4 = 240$.
$(x-2)^4$: coefficient of x^2 is $\frac{1}{2}4\cdot 3\cdot(-2)^2 = 24$.
$(x-2)^2$: coefficient of x^2 is $\frac{1}{2}2\cdot 1\cdot(-2)^0 = 1$.

The coefficient in question is then

$$1792 - 8\cdot 240 + 20\cdot 24 - 16 = \boxed{336}.$$

For **(b)** several high-school students gave the inductive argument given here. Denote by p_k the expression with k parentheses. The expression found above for p_3 shows that p_3 is a degree-8 polynomial in x, and so p_4 will be degree-16 and p_k will be degree-2^k. We are generating a sequence $\{p_1, p_2, p_3, \cdots\}$ of polynomials whose degree grows exponentially! Say the expression p_k with k parentheses, $k \geq 1$, has $p_{k,2}$ as the coefficient of x^2. We have shown $p_{3,2} = 336$. Likewise, let $p_{k,1}$ and $p_{k,0}$ be the coefficients of x^1 and x^0 in p_k. Furthermore,

$$p_{k+1} = (p_k - 2)^2 = p_k^2 - 4p_k + 4. \tag{6}$$

Now for $-4p_k$ the coefficients of x^2, x^1, and x^0 are, respectively, $-4p_{k,2}$, $-4p_{k,1}$, and $-4p_{k,0}$. Moreover, we know

$$\begin{aligned} p_k^2 &= \big((\text{terms in } x^3 \text{ or higher}) + p_{k,2}x^2 + p_{k,1}x + p_{k,0}\big)^2 \\ &= (\text{terms in } x^3 \text{ or higher}) + (2p_{k,2}p_{k,0} + p_{k,1}^2)x^2 + 2p_{k,1}p_{k,0}x + p_{k,0}^2. \end{aligned}$$

Therefore, (6) determines that

$$p_{k+1,0} = (p_{k,0}-2)^2, \qquad p_{k+1,1} = 2p_{k,1}(p_{k,0}-2), \qquad p_{k+1,2} = 2p_{k,2}(p_{k,0}-2) + p_{k,1}^2.$$

Clearly $p_{k,0} = 4$ for all k, and then $p_{k+1,1} = 4p_{k,1}$, where $p_{1,1} = -4$ is the coefficient of x in $(x-2)^2$. So $p_{2,1} = -16$, $p_{3,1} = -64$, and in general $p_{k,1} = -4^k$. The main recursion of interest is then the following.

$$\boxed{p_{k+1,2} = 4p_{k,2} + 4^{2k}}$$

Recursively,

$$\begin{aligned} p_{k+1,2} &= 4(4p_{k-1,2} + 4^{2(k-1)}) + 4^{2k} \\ &= 4^2 p_{k-1,2} + 4^{2k-1} + 4^{2k} \\ &= 4^3 p_{k-2,2} + 4^{2k-2} + 4^{2k-1} + 4^{2k}, \end{aligned}$$

or continuing

$$p_{k+1,2} = 4^{m+1} p_{k-m,2} + \sum_{\ell=0}^{m} 4^{2k-m+\ell}.$$

With the choice $m = k-1$,

$$p_{k+1,2} = 4^k p_{1,2} + \sum_{\ell=0}^{k-1} 4^{k+1+\ell} = 4^k + 4^{k+1} \sum_{\ell=0}^{k-1} 4^\ell = 4^k\left[1 + \tfrac{4}{3}(4^k - 1)\right] = \tfrac{1}{3}4^k(4^{k+1} - 1),$$

where we have used the summation formula for a geometric series. Let us confirm this result.

Claim: With the start value $p_{1,2} = 1$, the boxed two-term recursion is uniquely solved by

$$p_{k,2} = \tfrac{1}{3}4^{k-1}(4^k - 1).$$

Proof. The boxed equation has the form of a linear, inhomogeneous, constant-coefficient, first-order difference equation. Any sequence $\{p_{k,2}\}_{k=1}^{\infty}$ obeying the boxed equation is called a *solution* of the difference equation. It is uniquely determined by the start value (or initial condition) $p_{1,2}$. For an applied discussion of such equations, see W. Gautschi, *Numerical Analysis*, second edition (Birkhäuser, 2012). Here we are solving

$$p_{k+1,2} = 4p_{k,2} + 4^{2k}, \qquad p_{1,2} = 1.$$

The expression stated in the claim gives

$$p_{1,2} = \tfrac{1}{3}4^{1-1}(4^1 - 1) = \tfrac{1}{3}4^0 \cdot 3 = 1,$$

and so clearly obeys the initial condition. Moreover, using the proposed solution to compute the right-hand side of the recursion, we find

$$\begin{aligned} 4p_{k,2} + 4^{2k} &= 4\left[\tfrac{1}{3}4^{k-1}(4^k - 1)\right] + 4^{2k} \;=\; \tfrac{1}{3}4^k(4^k - 1) + 4^{2k} \;=\; \tfrac{1}{3}\left[4^k(4^k - 1) + 3\cdot 4^k 4^k\right] \\ &= \tfrac{1}{3}4^k(4^k - 1 + 3\cdot 4^k) \;=\; \tfrac{1}{3}4^k(4\cdot 4^k - 1) \;=\; \tfrac{1}{3}4^k(4^{k+1} - 1). \end{aligned}$$

This is the same expression with k replaced by $k+1$, establishing the inductive step. □

The above analysis shows that

$$p_{2025,2} = \tfrac{1}{3}4^{2024}(4^{2025} - 1).$$

This is a super large number! Indeed, since $\log_{10} 4 \simeq 0.6020599913279624$, one can show $p_{2025,2} \simeq \frac{1}{3}10^{2467603}$. This is larger than the largest machine number $2^{971}(2^{53} - 1) \simeq 1.7977 \times 10^{308}$ in Matlab which uses a 64-bit storage format for double precision. By way of comparison, the Eddington number, the number of protons in the observable universe, is estimated to be about 10^{80}.

UNM-PNM Round 2

Solutions

Sean Choi

March 2025

Solution 1.
We first find the maximum number of coins from which we are able to find the counterfeit coin with a single weighing. Clearly, if there are 2 coins, we are able to find the heavier one with 1 weighing.

What about 3? There are two cases. Either the scale is balanced, which implies that the coin not on the scale is the counterfeit one. Otherwise, the scale must not be balanced, which means that the counterfeit coin is the one which the scale indicates to be heavier. Thus, we are able to detect the counterfeit coin with a single weighing when there are 3 total coins.

Is this possible for 4 coins as well? Suppose we put only one coin on each side of the scale. In this case, there are two possibilities for the weighing. If the scale is unbalanced, then the coin which the scale indicates to be heavier is the counterfeit one. If the scale is balanced, we know that one of the two remaining coins is counterfeit. However, we cannot tell which one is the counterfeit one without another weighing.

What if we put 2 coins on each side? One side will always be heavier, but there is still no way to tell which of the two coins on the heavier side is the true counterfeit one.

Thus, if there less than or equal to 3 coins, we can find the counterfeit coin with a single weighing.

(a) Note that we can split the 9 coins into 3 groups of 3 coins. Then, we can weigh two of the groups of 3 coins. If they are equal, we know that the counterfeit coin is among the unweighed 3 coins. It only takes 1 more weighing to identify the counterfeit coin.

If the scale is unbalanced, the counterfeit coin is one of the 3 coins on the heavier side. Again, it only takes 1 more weighing to identify the counterfeit coin.

Thus, it takes only 2 weighings to identify the counterfeit coin, as desired.

(b) The answer is $\boxed{3}$. Note that we can split the 27 coins into 3 groups of 9 coins. Similarly as in the previous part, it takes only one weighing to identify which pile of 9 coins includes the counterfeit coin. Then, we know that it takes 2 weighings to identify the counterfeit coin amongst 9 coins, so the answer is $1 + 2 = 3$.

We must show that this is indeed the minimum. It suffices to show that we cannot find the counterfeit coin amongst 10 coins with less than 3 weighings. We can split 10 coins into 3 piles as 5-5-0, 4-4-2, 3-3-4, 2-2-6, or 1-1-8. If we take the first two piles of coins in each of these cases, there is always the possibility that we must find the counterfeit coin amongst at least 4 coins. However, we found above that this requires more than 1 weighing. Thus, it is not possible to always find the counterfeit coin with 2 weighings. Thus, 3 is indeed the minimum for 27 coins.

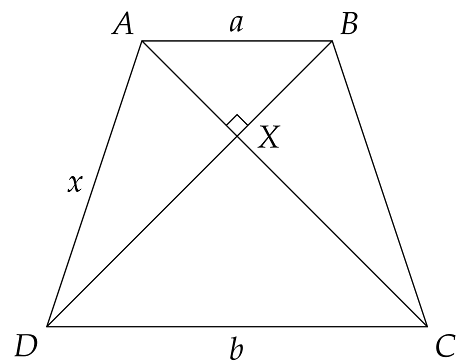
Solution 2. This is a stars and bars problem: it can be likened to arranging M stars and $N - 1$ bars (or dividers). For example, putting all 3 quanta in the first oscillator can be represented as ***| | |, where the bars divide the stars into the oscillators, and where the stars represent the quanta. There are a total of 6 spots, and we can choose 3 of them to be the bars. Thus, the answer is $\binom{6}{3} = \boxed{20}$.

Solution 3. Let $x = \sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$. Then, we have

$$\begin{aligned} x^2 &= 3 + 2\sqrt{2} + 3 - 2\sqrt{2} - 2\sqrt{(3 + 2\sqrt{2})(3 - 2\sqrt{2})} \\ &= 6 - 2\sqrt{3^2 - (2\sqrt{2})^2} \\ &= 6 - 2\sqrt{9 - 8} \\ &= 6 - 2\sqrt{1} \\ &= 4. \end{aligned}$$

Thus, we have $x = 2$ or $x = -2$. However, clearly $\sqrt{3 + 2\sqrt{2}} > \sqrt{3 - 2\sqrt{2}}$ which implies that $x > 0$. Taking the positive solution for x yields $x = \boxed{2}$.

Solution 4. We solve a related general version of this problem. Suppose we have isosceles trapezoid $ABCD$ such that AB is parallel to CD. Additionally, suppose that the diagonals are perpendicular and that we have $AB = a$ and $CD = b$. We wish to find the length of $AD = BC = x$. This setup looks like the following:



Note that this is a similar setup as the original, except that we are expressing AD in terms of AB and CD, instead of finding CD from AD and AB. This will still allow us to use the resulting expression for x in terms of a and b to find CD.

Because the trapezoid is isosceles, we have $\triangle ACD \cong \triangle BDC$. This means that $\angle DAC = \angle CBD$. Then, $\triangle AXD \cong \triangle BXC$ since $AD = BC$, $\angle DAX = \angle CBX$, and since $\angle AXD = \angle BXC$. This implies that $AX = BX$ and that $DX = CX$.

In other words, both $\triangle AXB$ and $\triangle CXD$ are 45-45-90 triangles. This means that $AX = \frac{a}{\sqrt{2}}$ and $DX = \frac{b}{\sqrt{2}}$. Then, by the Pythagorean Theorem,

$$x^2 = AX^2 + DX^2 = \frac{a^2}{2} + \frac{b^2}{2} = \frac{a^2 + b^2}{2}.$$

Thus,

$$x = \sqrt{\frac{a^2 + b^2}{2}}.$$

In our problem, however, we are given $x = 5$, $a = 1$, and we wish to find b. Using the property which we derived above, we have

$$5 = \sqrt{\frac{1^2 + b^2}{2}} \implies b = \boxed{7}.$$

Solution 5.
(a) The terms a_1, a_2, and a_3 form an arithmetic progression. This means that

$$a_2 - a_1 = a_3 - a_2.$$

This is because the difference between consecutive terms is always constant, so the difference between the first and second terms must equal the difference between the third and second terms. Substituting values, we have

$$\cos(x) - \tan(x) = \sec(x) - \cos(x).$$

Rearranging yields

$$2\cos(x) = \sec(x) + \tan(x).$$

Now, we know that $\sec(x) = \frac{1}{\cos(x)}$ and $\tan(x) = \frac{\sin(x)}{\cos(x)}$, so we have

$$2\cos(x) = \frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)} = \frac{1+\sin(x)}{\cos(x)}.$$

Since we are given that the arithmetic progression consists of positive numbers, we can multiply both sides of our equation by $\cos(x)$ which yields

$$2\cos^2(x) = 1 + \sin(x).$$

Now, the Pythagorean Identity gives us $\sin^2(x) + \cos^2(x) = 1$. Isolating $\cos^2(x)$ yields $\cos^2(x) = 1 - \sin^2(x)$. Substituting into our equation, we have

$$2(1 - \sin^2(x)) = 1 + \sin(x).$$

Expanding and rearranging yields

$$2\sin^2(x) + \sin(x) - 1 = 0.$$

This quadratic in $\sin^2(x)$ factors as $(2\sin(x) - 1)(\sin(x) + 1)$, which means that either $\sin(x) = \frac{1}{2}$ or that $\sin(x) = -1$. Since we are given that x is in the first quadrant, we must have $\sin(x) = \frac{1}{2}$. Thus, $x = \boxed{30^\circ} = \frac{\pi}{6}$.

(b) To find r, the common difference, it suffices to compute $\cos(x) - \tan(x)$. We have

$$\begin{aligned} r &= \cos(30^\circ) - \tan(30^\circ) \\ &= \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{3} \\ &= \boxed{\frac{\sqrt{3}}{6}}. \end{aligned}$$

(c) Suppose that $\cot(x)$ is the nth term in the sequence. This means that $\cot(x) = a + (n-1)r$ where a is the first term and r is the common difference. We know that the first term is $\tan(x)$, and we also know that $r = \frac{\sqrt{3}}{6}$. Thus, we have

$$\cot(x) = \tan(x) + (n-1)\cdot\frac{\sqrt{3}}{6}.$$

We also know that $x = 30^\circ$, which means that $\cot(x) = \sqrt{3}$ and $\tan(x) = \frac{1}{\sqrt{3}}$. Then, we have

$$\sqrt{3} = \frac{1}{\sqrt{3}} + (n-1) \cdot \frac{\sqrt{3}}{6}.$$

Then,

$$\sqrt{3} = \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{6} \cdot n - \frac{\sqrt{3}}{6}.$$

Dividing by $\sqrt{3}$ yields

$$1 = \frac{1}{3} + \frac{1}{6} \cdot n - \frac{1}{6}.$$

This means that $n = 5$, so $\cot(x)$ occupies the fifth position in the sequence.

Solution 6. We complimentary count. First, there are a total of $\binom{32}{1}\binom{32}{1}$ ways to choose one black square and one white square.

Now, we count the number of invalid ways. That is, we count the number of ways to choose a black and white square in the same row or column. Note that there are no cases where we can select a white and black square in the same row **and** column.

There are 8 rows and 8 columns, which mean that we can choose any of 16 rows or columns to select the squares. In each row or column, there are 4 black squares and 4 white squares. So, there are $\binom{4}{1}\binom{4}{1}$ ways to choose a white and black square in each row. Thus, there are a total of $16 \cdot \binom{4}{1}\binom{4}{1}$ ways to choose a black and white square such that they lie on the same row or column.

Thus, the number of ways to select a black square and white square so that the chosen squares do not lie in the same row or column is

$$\binom{32}{1}\binom{32}{1} - 16\binom{4}{1}\binom{4}{1} = 32^2 - 16 \cdot 4^2 = \boxed{768 \text{ ways}}.$$

Solution 7.
(a) Let $x = \cos\left(\frac{\pi}{5}\right)$ and let $y = \cos\left(\frac{2\pi}{5}\right)$. By the double angle formula, we have $y = 2x^2 - 1$ (1). In addition, note that $-x = \cos\left(\frac{4\pi}{5}\right)$. Thus, we can use the double angle formula again to get $-x = 2y^2 - 1$ (2). If we subtract the two equations from each other, we get

$$x + y = 2x^2 - 2y^2,$$

which simplifies to

$$x + y = 2(x-y)(x+y).$$

Clearly, x and y are both positive, so we can divide through by $x + y$. We are then left with $x - y = \frac{1}{2}$.

Now, we substitute $y = x - \frac{1}{2}$ back into equation (1). This yields

$$2x^2 - x - \tfrac{1}{2} = 0.$$

By the quadratic formula and taking the positive solution, we have

$$\cos\left(\frac{\pi}{5}\right) = x = \frac{1+\sqrt{5}}{4}.$$

Similarly, we can substitute $x = \frac{1}{2} + y$ into equation (2) to get

$$2y^2 + y - \tfrac{1}{2} = 0.$$

Taking the positive solution yields

$$\cos\left(\frac{2\pi}{5}\right) = y = \frac{\sqrt{5}-1}{4},$$

and we are done.

(b) Let A_1 be the area of the hexagon and let A_2 be the area of the pentagon.

The general area of a regular polygon with n sides is

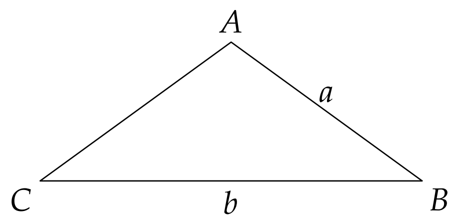
$$A = \frac{s^2 \cdot n}{4\tan\left(\frac{\pi}{n}\right)}.$$

This is because we can split our regular polygon into n congruent triangles by connecting each vertex to the center of the polygon. Then, we can drop a perpendicular to each side of the polygon from the center. The area of each of the n congruent triangles is then $\frac{s}{2} \cdot \frac{s}{2\tan\left(\frac{\pi}{n}\right)}$. Multiplying this area by n for each of the n congruent triangles yields the desired formula.

Thus, we have

$$\frac{A_1}{A_2} = \frac{\frac{6b^2}{4\tan(\pi/6)}}{\frac{5a^2}{4\tan(\pi/5)}} = \frac{6b^2\tan(\pi/5)}{5a^2\tan(\pi/6)}.$$

Now, consider the triangle portion that is formed by the hexagon and the pentagon:



Because $\angle BAC$ is one of the angles of the regular pentagon, it has measure $3\pi/5$. Then, $\angle ACB = \angle ABC = \pi/5$. Thus, we have

$$\cos\left(\frac{\pi}{5}\right) = \frac{b/2}{a}.$$

This means that

$$\frac{b^2}{a^2} = 4\cos^2\left(\frac{\pi}{5}\right).$$

So,

$$\begin{aligned}\frac{A_1}{A_2} &= \frac{6b^2\tan(\pi/5)}{5a^2\tan(\pi/6)} = \frac{24\cos^2(\pi/5)\tan(\pi/5)}{5\tan(\pi/6)}\\ &= \frac{24\cos(\pi/5)\sin(\pi/5)}{5\tan(\pi/6)}.\end{aligned}$$

Since $\tan(\pi/6) = \frac{1}{\sqrt{3}}$, we have

$$\frac{A_1}{A_2} = \frac{24\sqrt{3}\cos(\pi/5)\sin(\pi/5)}{5}.$$

Note that $\sin(2\pi/5) = 2\cos(\pi/5)\sin(\pi/5)$, so

$$\frac{A_1}{A_2} = \frac{12\sqrt{3}\sin(2\pi/5)}{5}.$$

From part (a), we found that $\cos\left(\frac{2\pi}{5}\right) = \frac{\sqrt{5}-1}{4}$. Since $\sin^2(\theta) + \cos^2(\theta) = 1$, we have

$$\sin\left(\frac{2\pi}{5}\right) = \sqrt{\frac{5+\sqrt{5}}{8}}.$$

Thus

$$\frac{A_1}{A_2} = \frac{12\sqrt{3}}{5}\cdot\sqrt{\frac{5+\sqrt{5}}{8}} = \boxed{\frac{3}{5}\sqrt{6(5+\sqrt{5})}}.$$

Solution 8. Let P_k denote the entire expression when there are k pairs of parentheses. Let $p_{k,i}$ denote the coefficient of x^i in P_k.

Note that $P_1 = (x-2)^2$ and $P_{k+1} = (P_k - 2)^2 = P_k^2 - 4P_k + 4$. Now, note that when we expand this expression, the only terms we need to consider are those with degree 2 or less, since terms with degree greater than 2 will not contribute to the coefficient of x^2. We have

$$\begin{aligned} P_{k+1} &= ((\text{unnecessary terms}) + p_{k,2}x^2 + p_{k,1}x + p_{k,0})^2 \\ &\quad - 4((\text{unnecessary terms}) + p_{k,2}x^2 + p_{k,1}x + p_{k,0}) + 4 \\ &= (\text{unnecessary terms})^2 + (\text{more unnecessary terms}) \\ &\quad + 2p_{k,2}p_{k,0}x^2 + p_{k,1}^2x^2 - 4p_{k,2}x^2 + (\text{unnecessary constant}). \end{aligned}$$

So,

$$p_{k+1,2} = 2p_{k,2}p_{k,0} + p_{k,1}^2 - 4p_{k,2}.$$

Now, note that $p_{k,0} = 4$ for all k since we always subtract 2 and square. Using a similar expansion method as above, we can find that $p_{k+1,1} = 2p_{k,1}p_{k,0} - 4p_{k,1} = 4p_{k,1}$. Since $p_{1,1} = -4$, we have $p_{k,1} = -4^k$. Thus,

$$p_{k+1,2} = 8p_{k,2} + 4^{2k} - 4p_{k,2} = 4p_{k,2} + 4^{2k}.$$

Now, note that

$$\begin{aligned} p_{k+1,2} &= 4p_{k,2} + 4^{2k} = 4(4p_{k-1,2} + 4^{2k-1}) + 4^{2k} \\ &= 4(4(4p_{k-2,2} + 4^{2k-4}) + 4^{2k-2}) + 4^{2k} \\ &= 4^{m+1}p_{k-m,2} + \sum_{i=2k-m}^{2k} 4^i \end{aligned}$$

Taking $m = k-1$ yields

$$p_{k+1,2} = 4^k p_{1,2} + \sum_{i=k+1}^{2k} 4^i = 4^k + \frac{4^{2k+1} - 4^{k+1}}{3} = 4^k\left(\frac{3 + 4^{k+1} - 4}{3}\right) = \frac{4^{2k+1} - 4^k}{3}.$$

We can plug this closed formula into the recurrence to verify that it is correct, which it turns out to be. Thus,

$$p_{k,2} = \frac{4^{2k-1} - 4^{k-1}}{3} = \boxed{\frac{16^k - 4^k}{12}}.$$

From here, the answers for both parts can be easily calculated.