UNM-PNM Statewide High School Mathematics Contest 2024-2025
Round-1 Problem Solutions

Dear Students,
If you have suggestions about the Contest, or if you have different solutions to any of this year's first round problems, please mail them to:

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Please remember that you can find information about past contests at `http://mathcontest.unm.edu`

We express our gratitude to Coach Sean Choi for the time spent with students in Albuquerque and Los Alamos, and for running an online solution session for the first round exam. We also thank Sean, Bill Cordwell, and Laszlo Zolyomi for sharing their solutions with us. Finally, thanks to all participants, their teachers, and families. You are an inspiration for us!

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1. ENTERED ANSWER: $\boxed{2025}$

Solution 1. You can verify that $\frac{2024}{2025}$ is the larger fraction by cross-multiplication and comparison. Comparison of $\frac{2023}{2024}$ and $\frac{2024}{2025}$ is equivalent to comparison of 2023×2025 and $2024 \times 2024 = 2024^2$. Direct calculation shows $2023 \times 2025 = 4,096,575 < 4,096,576 = 2024^2$, with the conclusion $\frac{2023}{2024} < \frac{2024}{2025}$.

Solution 2. The path in **Solution 1** requires a calculator, or a willingness to perform two burdensome multiplications by hand. Alternatively, notice that

$$2023 \times 2025 = (2024 - 1) \times (2024 + 1) = 2024^2 - 1 < 2024^2.$$

Rearrangement of $2023 \times 2025 < 2024^2$ gives $\frac{2023}{2024} < \frac{2024}{2025}$. This calculation works for any three consecutive whole numbers. For this problem $n - 1 = 2023$, $n = 2024$, and $n + 1 = 2025$, but we may consider any positive integer $n \geqslant 1$. If asked to compare $\frac{n-1}{n}$ and $\frac{n}{n+1}$, we cross multiply and compare $(n-1)(n+1)$ to n^2. Notice that

$$(n-1)(n+1) = n^2 - 1 < n^2,$$

and so $\frac{n-1}{n} < \frac{n}{n+1}$.

Solution 3 (due to Bill Cordwell and Laszlo Zolyomi). Notice that $\frac{n}{n+1} = 1 - \frac{1}{n+1}$ is increasing and approaching 1 as $n \to \infty$, so $\frac{2023}{2024} < \frac{2024}{2025}$. One might also simply note that

$$\frac{2023}{2024} = 1 - \frac{1}{2024} < 1 - \frac{1}{2025} = \frac{2024}{2025}.$$

Remark. The third solution asserts that $n/(n+1)$ is increasing (in fact, on integers $n \geqslant 0$). Similarly, the second solution has shown the equivalent result that $(n-1)/n$ is increasing on integers $n \geqslant 1$. We can further establish the result for positive real numbers at least as large as 1 by manipulation of inequalities. Start with positive real numbers, x_1 and x_2, obeying

$$1 \leqslant x_1 < x_2,$$

which upon reciprocation gives

$$\frac{1}{x_2} < \frac{1}{x_1} \leqslant 1,$$

and upon multiplication by -1,

$$-1 \leqslant -\frac{1}{x_1} < -\frac{1}{x_2}.$$

Addition of 1 to each term in the chain yields

$$0 \leqslant 1 - \frac{1}{x_1} < 1 - \frac{1}{x_2},$$

that is

$$0 \leqslant \frac{x_1 - 1}{x_1} < \frac{x_2 - 1}{x_2}.$$

With $f(x) = (x-1)/x$, the last inequality shows that $1 \leqslant x_1 < x_2$ implies $0 \leqslant f(x_1) < f(x_2)$. We conclude that $f(x)$ is an increasing function on the half-open interval $[1, \infty)$. With this result established, we then have $f(n+1) > f(n)$ for any integer $n \in [1, \infty)$.

2. ENTERED ANSWER: $\boxed{9900}$

Preface. The problem asserts that there are 24 four-digit numbers, where the digits 2,4,5,7 appear exactly once. One can of course list all these numbers and count them, to verify that the assertion is correct. However, a better way of counting is to consider four place holders ␣ ␣ ␣ ␣ for the digits. In the first holder (say the leftmost) you can choose any of the 4 numbers, in the second holder you can choose among the 3 numbers that have not yet been chosen, in the third holder you can choose from among the 2 numbers left, and in the fourth holder there is just 1 number left that should go there. Altogether, there are $4 \times 3 \times 2 \times 1 = 4! = 24$ possibilities. The advantage in this way of thinking would be evident were you asked, for example, "how many nine-digit numbers you can write with the digits $1, 2, 3, 4, 5, 6, 7, 8, 9$?" By a similar reasoning you would conclude that the answer is $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 9! = 362,880$. For this example you certainly don't want to write out all the possible nine-digit numbers and count them!

Solution 1. The largest such number is 7542 and the smallest is 2457. The first number cannot begin with 4, 5, or 7, since $2 \times 4000 = 8000$, already larger than 7542. So our number must begin with 2 and there are six possibilities: 2754, 2745, 2574, 2547, 2475, 2457. Moreover, the multiplier in question must be either 2 or 3, since 4 times any of these numbers is greater than 7542. Indeed, notice that $4 \times 2400 = 9600$ already. If the multiplier were 2, then the first number would have to be either 2547 or 2457, since otherwise we would generate an 8 or 0 as the last digit. However, $2 \times 2547 = 5094$ and $2 \times 2457 = 4914$. So these won't work. Our multiplier must then be 3. In this case $2754, 2745, 2574, 2547$ are all too big. Indeed, $3 \times 2547 = 7641$. So our first number must be either 2475 or 2457. We check that $7425 = 3 \times 2475$. The reported answer is then $7425 + 2475 = 9900$.

Solution 2 (due to Bill Cordwell). The largest such number is 7542 and the smallest is 2457. The leftmost digit of the pair's larger number cannot be a multiple of more than 3, else, since $4 \times 2000 = 8000 > 7542$, the pair's smaller number would be less than 2457. We first look at multiples of 3. Because 3×2457 is already larger than 7000, the pair's larger number must have 7 as its leftmost digit. So we look at 7542, which would correspond to 2514 as the smaller number (nope). 7524 does not work, nor does 7452, but $7425 = 3 \times 2475$.

3. ENTERED ANSWER: $\boxed{1}$

Preface. Even before trying to solve the problem, one might consider variations of the game. What would happen if you had more piles? Or if the two piles had different numbers of stones? Or if it's the loser who is the last to draw stones? In such variations is there a winning strategy for one of the players? As pointed out by Sean Choi, these are all examples of the *Game of Nim* (see `https://en.wikipedia.org/wiki/Nim`).

Solution. The reported answer is 1, since Cora can always win if she plays wisely. To see why, let's explore how the game might unfold. We note that if, later in the game, a player is confronted with two piles, each with a single stone, then their opponent will win on the next turn. Indeed, when confronted with two single-stone piles, all a player can do is choose one or the other stone, leaving behind one single-stone pile for their opponent to grab.

We argue that were the game played with two piles of 7 stones, then the person (player 2) whose first turn comes second can always win, that is the person (player 1) who starts the game will always lose if the second player plays wisely. Let's explore why with examples.

```
       Game Scenario 1

  pile1      pile2
 *******    *******  begin with two full piles
            *******  player1 just took 7 stones from pile1
                     player2 just took 7 stones from pile2 and won

       Game Scenario 2

  pile1      pile2
 *******    *******  begin with two full piles
       *    *******  player1 just took 6 stones from pile1
       *          *  player2 just took 6 stones from pile2
                  *  player1 just took 1 stone  from pile1
                     player2 just took 1 stone  from pile2 and won

       Game Scenario 3

  pile1      pile2
 *******    *******  begin with two full piles
      **    *******  player1 just took 5 stones from pile1
      **         **  player2 just took 5 stones from pile2
       *         **  player1 just took 1 stone  from pile1 (forced hand)
       *          *  player2 just took 1 stone  from pile2
                  *  player1 just took 1 stone  from pile1
                     player2 just took 1 stone  from pile2 and won
```

Note that in Scenario 3, when player 1 is confronted with two piles, each with two stones, they must choose only a single stone from one of the piles. If they were to grab a whole pile (two stones) they would lose on the next turn. So their *hand is forced*. These scenarios suggest that by always choosing the same number of stones as drawn previously by their opponent, but from the opposite pile, player 2 can always win the two-pile game. Let's formalize the argument.

We refer to a turn by player 1 followed by a turn by player 2 as a *round* of the game. Assume that, at the start of a round, player 1 is confronted with two piles of p stones. Necessarily for our game $1 \leqslant p \leqslant 7$, but our argument works when $1 \leqslant p \leqslant N$ for any whole number $N > 1$ (that is, for a game of two piles with N stones each). If $p = 1$, then player 1 has before them a losing proposition. The rules of the game demand that they choose a stone from one of the piles, leaving behind a single-stone pile that player 2 will grab and win. Therefore, let us assume that $2 \leqslant p \leqslant N$. Player 1 cannot grab all the stones in one pile, or player 2 will win on their next turn. To forestall losing immediately, player 1 must then take k stones from one of the piles, with $1 \leqslant k \leqslant p - 1$. Due to symmetry, it does not matter which of the p-stone piles these k stones are taken from. Now at the start of their turn, player 2 is confronted with two piles, one with p stones and one with $p-k$ stones, where $1 \leqslant p-k \leqslant p-1$. Player 2 then wisely takes k stones from the larger pile, and a new round begins. At the start of the new round, player 1 is confronted with two piles of $p - k \in \{1, \ldots, p-1\}$ stones, that is the new round starts with precisely the same scenario as the previous round, but with smaller piles. Successive rounds as described in this paragraph will then inevitably lead to round in which player 1 starts off confronted with two single-stone piles, the losing proposition.

Cora's strategy is clear. She begins be taking all stones from one pile. This play effectively turns the three-pile game into a two-pile game, and one in which the roles of player 1 and player 2 are reversed. Cora is now in the winning player-2 role for the two-pile game.

4. ENTERED ANSWER: $\boxed{3}$

Preface. Since the halcyon age of Greek mathematics circa 300 BC, people have known that a regular hexagon can be circumscribed by a circle. Further, given the circumscribing circle (known from its center and radius), the inscribed hexagon can be constructed using a compass and a straight-edge. From this construction follows a fact that we assert: *the line segments joining the opposite vertices of a regular hexagon divide it into 6 equilateral triangles, each one congruent to the others.*

Solution 1. The area of the large hexagon is the area of 6 equilateral triangles of side length b, that is $A_{\text{large}} = 6 \cdot \sqrt{3}b^2/4 = 3\sqrt{3}b^2/2$. Likewise, the area of the small hexagon is $A_{\text{small}} = 3\sqrt{3}a^2/2$, so

$$\frac{A_{\text{large}}}{A_{\text{small}}} = \left(\frac{b}{a}\right)^2.$$

Note we may obtain this result simply by knowing that the area of an equilaterial triangle is proportional to its squared side length, *without knowing the constant of proportionality.* However, from the diagram $\frac{1}{2}b = a\cos(\pi/6)$, so $b = 2a\cos(\pi/6) = a\sqrt{3}$. The ratio is therefore

$$\frac{A_{\text{large}}}{A_{\text{small}}} = 3.$$

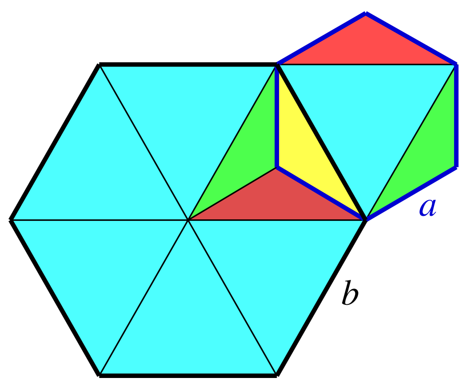
Solution 2. The larger hexagon is subdivided into 6 equilateral triangles shown in the figure. Five of these are colored cyan (light blue) and the sixth has itself been subdivided into three, smaller and isosceles, triangles which are colored yellow, red, and green. Each of these smaller triangles has base length b, with an opposite angle of 120° (the other two angles are both 30°). These small triangle are then congruent. Denote by α the area of one of these smaller 30°-30°-120° triangles. Each of the equilateral triangles then has area 3α, and the area of the larger hexagon is

$$A_{\text{large}} = 6(3\alpha) = 18\alpha.$$

Now turn to the smaller hexagon. It is subdivided into four triangles. The first, the interior triangle colored cyan, is an equilateral triangle congruent to one of the six which subdivided the larger hexagon. The remaining three are smaller isosceles triangles of area α. Indeed each is congruent to one of the smaller triangles depicted within the larger hexagon: the two red triangles are congruent, the two green triangles are congruent, and the yellow triangle is the overlap (intersection) of the larger hexagon and the smaller hexagon. Therefore, the area of the smaller hexagon is

$$A_{\text{small}} = \underbrace{3\alpha}_{\substack{\text{yellow}\\ \text{red}\\ \text{green}}} + \underbrace{3\alpha}_{\text{cyan}} = 6\alpha.$$

We then know the sought ratio $A_{\text{large}}/A_{\text{small}} = 3$ of the areas. For our solution we need not know α, but as $\frac{1}{3}$ the area of an equilaterial triangle of side-length b, we may easily calculate that $\alpha = b^2/(4\sqrt{3})$. From this result $A_{\text{large}} = \frac{3}{2}\sqrt{3}b^2$, and $A_{\text{small}} = \frac{1}{2}\sqrt{3}b^2$. Since in terms of a and b respectively, the areas of A_{small} and A_{large} must have the same expressions, we see that $b = a\sqrt{3}$.



5. ENTERED ANSWER: $\boxed{3}$

Solution. This is somewhat of a trick question, since we need to notice that

$$x = \sqrt{1 + \sqrt{1 + \sqrt{1}}} = \sqrt{1 + \sqrt{2}},$$

and should start with this recognition before embarking on the calculation. With this simplification in hand,

$$x^2 = 1 + \sqrt{2},$$

and

$$x^4 = (1 + \sqrt{2})^2 = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}.$$

Whence $x^4 - 2\sqrt{2} = 3$.

6. ENTERED ANSWER: $\boxed{4321}$

Solution. We are told that the friend with the blue hair is neither Mario nor Andrea and is standing between Jess and the friend with the purple hair. We know then that neither Mario nor Andrea have blue hair. We also know that Jess is standing next to the friend with the blue hair and she cannot have purple or blue hair, hence she must have either pink or green hair. Note that at this point all but Paul cannot have blue hair, so **Paul must be the one with the blue hair**, and green, pink, purple are the colors left to assign. We are also told that the friend with pink hair is standing between Mario and the friend with the green hair. Therefore, Mario does not have pink or green hair; at this point, we then know that **Mario has purple hair**, but let's pretend that we have not made this observation.
If Jess has pink hair, then she is standing between Mario and the friend with the green hair, but since she is standing next to the friend with blue hair, it must be that Mario has blue hair, which is false. Therefore, Jess does not have pink hair. It must be that **Jess has green hair**. At this point, we know that Mario and Andrea have not dyed their hair blue or green, but we already know Mario's hair is not pink, so **Mario has purple hair**, and Andrea does not have purple hair. **Andrea must have pink hair**. The filled-out table follows.

name	Mario	Andrea	Jess	Paul
color	purple	pink	green	blue
color number	4	3	2	1

The reported answer is therefore 4321.

7. ENTERED ANSWER: $\boxed{4}$

Solution. Let x denote the life span of the wild turkey, what we are asked for. Denote by W, b, g, w, t, and e the life spans of the bowhead whale (W), brown bear (b), western gorilla (g), wolverine (w), tiger (t), and elephant (e). Finally, let $S = x + W + b + g + w + t + e$ denote the sum of the life spans of all 7 animals. With this notation, let us now translate the given information in equations. We are told:

$$\begin{aligned} W &= 8 \times b, \\ b &= g - 11, \\ g &= 3 \times w, \\ w &= x + 8, \\ x &= t - 10, \\ t &= \frac{e}{4}, \end{aligned} \tag{1}$$

and finally that

$$e = \frac{S - 11}{6} = \frac{x + W + b + g + w + t + e - 11}{6}. \tag{2}$$

Our job is to find x. This is a system of 7 linear equations in 7 variables, and, as it turns out, this particular system has exactly one solution.

Let us express the life span of each animal in terms of x, the life span of the wild turkey. We first work upward starting with the fourth equation in (1),

$$\begin{aligned} w &= x + 8 \\ g &= 3w = 3x + 24 \\ b &= g - 11 = 3x + 13 \\ W &= 8b = 24x + 104. \end{aligned}$$

Then from the fifth and sixth equation in (1),

$$\begin{aligned} t &= x + 10 \\ e &= 4t = 4x + 40. \end{aligned}$$

Using these formulas, we may write the sum S solely in terms of x. Indeed,

$$\begin{aligned} S &= x + (24x + 104) + (3x + 13) + (3x + 24) + (x + 8) + (x + 10) + (4x + 40) \\ &= 37x + 199. \end{aligned}$$

We now substitute $e = 4x + 40$ and $S = 37x + 199$ into $e = \frac{1}{6}(S - 11)$ from (2), and then solve for x. The substitution yields

$$\begin{aligned} 4x + 40 &= \tfrac{1}{6}(37x + 199 - 11) \\ &= \tfrac{1}{6}(37x + 188). \end{aligned}$$

Multiply both sides of the equation by 6 to reach $24x + 240 = 37x + 188$. Now subtract $24x$ and 188 from both sides of the equation to get $240 - 188 = 37x - 24x$, or equivalently $52 = 13x$, determining that $x = 4$. The average span life of the wild turkey is 4 years! Once you know x, you can find the average life span of all the other animals: $w = 12$, $g = 36$, $b = 25$, $W = 200$, $t = 14$, and $e = 56$.

Hope you had a wonderful Thanksgiving and are getting ready for the upcoming Holidays!

Alternate solution. Here we employ the technique of *Gaussian elimination* to solve the system of equations described above. The technique is most effectively wielded using the notation of augmented matrices, but here we will work with the equations themselves. As described above, our system is

$$\begin{array}{ccccccccc} W & -8b & & & & & & = & 0 \\ & b & -g & & & & & = & -11 \\ & & g & -3w & & & & = & 0 \\ & & & w & -x & & & = & 8 \\ & & & & x & -t & & = & -10 \\ & & & & & t & -\frac{1}{4}e & = & 0 \\ W & +b & +g & +w & +x & +t & -5e & = & 11 \end{array} \tag{3}$$

To get the last equation in this list, we have rearranged the last equation

$$e = \tfrac{1}{6}(-11 + W + b + g + w + x + t + e)$$

stated in the problem. Gaussian elimination is a process whereby we put the system into an *upper triangular form* suitable for solving. Label the equations in (3) and in the subsequent systems written below as E1, E2, E3, E4, E5, E6, and E7. That is E1 is the first equation, and E7 is the last one. We now begin the elimination process. First, in the last system replace E7 by [E7 minus E1].

$$\begin{array}{rrrrrrcr} W & -8b & & & & & = & 0 \\ & b & -g & & & & = & -11 \\ & & g & -3w & & & = & 0 \\ & & & w & -x & & = & 8 \\ & & & & x & -t & = & -10 \\ & & & & & t \;\; -\frac{1}{4}e & = & 0 \\ & 9b & +g & +w & +x & +t \;\; -5e & = & 11 \end{array}$$

Next, in the last system replace E7 by [E7 minus 9×E2].

$$\begin{array}{rrrrrrcr} W & -8b & & & & & = & 0 \\ & b & -g & & & & = & -11 \\ & & g & -3w & & & = & 0 \\ & & & w & -x & & = & 8 \\ & & & & x & -t & = & -10 \\ & & & & & t \;\; -\frac{1}{4}e & = & 0 \\ & & 10g & +w & +x & +t \;\; -5e & = & 110 \end{array}$$

Next, in the last system replace E7 by [E7 minus 10×E3].

$$\begin{array}{rrrrrrcr} W & -8b & & & & & = & 0 \\ & b & -g & & & & = & -11 \\ & & g & -3w & & & = & 0 \\ & & & w & -x & & = & 8 \\ & & & & x & -t & = & -10 \\ & & & & & t \;\; -\frac{1}{4}e & = & 0 \\ & & & 31w & +x & +t \;\; -5e & = & 110 \end{array}$$

Next, in the last system replace E7 by [E7 minus 31×E4].

$$\begin{array}{rrrrrrcr} W & -8b & & & & & = & 0 \\ & b & -g & & & & = & -11 \\ & & g & -3w & & & = & 0 \\ & & & w & -x & & = & 8 \\ & & & & x & -t & = & -10 \\ & & & & & t \;\; -\frac{1}{4}e & = & 0 \\ & & & & 32x & +t \;\; -5e & = & -138 \end{array}$$

Next, in the last system replace E7 by [E7 minus 32×E5].

$$\begin{array}{rrrrrrcr} W & -8b & & & & & = & 0 \\ & b & -g & & & & = & -11 \\ & & g & -3w & & & = & 0 \\ & & & w & -x & & = & 8 \\ & & & & x & -t & = & -10 \\ & & & & & t \;\; -\frac{1}{4}e & = & 0 \\ & & & & & 33t \;\; -5e & = & 182 \end{array}$$

Next, in the last system replace E7 by [E7 minus 33×E6].

$$\begin{array}{ccccccccc}
W & -8b & & & & & & = & 0\\
& b & -g & & & & & = & -11\\
& & g & -3w & & & & = & 0\\
& & & w & -x & & & = & 8\\
& & & & x & -t & & = & -10\\
& & & & & t & -\frac{1}{4}e & = & 0\\
& & & & & & \frac{13}{4}e & = & 182
\end{array}$$

Now we are ready to solve using a process known as *backward substitution.* The last equation tells us that $e = \frac{1}{13}728 = 56$, and then, working bottom-to-top, that $t = \frac{1}{4}e = 14$ and $x = t - 10 = 4$. We could stop here since we have x, or continue to solve for the rest of the variables. Proceeding, $w = x + 8 = 12$, $g = 3w = 36$, $b = g - 11 = 25$, and, finally, $W = 8b = 200$.

If you know linear algebra you could have represented the system with a 7×7 array of numbers (a matrix) and a 7×1 array (a vector inhomogeneity), and used matrix algebra techniques to find the solution. Elimination is one strategy, but there are others, for example, QR-factorization. In this case these elaborate strategies are not necessary, but if you have more complicated linear relations between the variables, and many more variables, then this is the way to go. You may also enlist the help of a computer. In real life you may need to solve a linear systems with hundreds, thousands, millions, or even billions of variables.

8. ENTERED ANSWER: $\boxed{22}$

Solution 1 (due to Bill Cordwell). For convenience we take $n = k+6$, where $k \in [-6, \infty)$. Then $n^2-11n+25 = k^2+k-5 = f(k)$. Note that if $k > 5$, then $(k+1)^2 > k^2+k-5 > k^2$, in which case $f(k)$ would not be a perfect square. At this point, the possibilities are few enough that we can just try them. Note that $k = -6$ and $k = 5$ give the values $f(k) = 25$, which is a perfect square. Moreover, $f(-3) = f(2) = 1$, which are perfect squares also. These are the only choices of $k \in \{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ that work. Conversion back to n gives $n = 0$, 3, 8, and 11. The sum is 22.

Solution 2 (expanded account of the argument made by Sean Choi). We seek integers $n \geqslant 0$ such that $m^2 = n^2 - 11n + 25$ is a perfect square, and first rule out values of n which will not work. Notice that $n > 11$ implies $n + 25 > 36$. Therefore, assuming $n > 11$, we have

$$(n-6)^2 = n^2 - 12n + 36 < n^2 - 12n + (n + 25) = n^2 - 11n + 25.$$

Moreover, for any $n > 0$,

$$n^2 - 11n + 25 = n^2 - 10n + 25 - n < n^2 - 10n + 25 = (n-5)^2.$$

Therefore, subject to $n > 11$, combination of these two inequalities gives

$$(n-6)^2 < n^2 - 11n + 25 < (n-5)^2,$$

showing that $n^2 - 11n + 25$ lies between consecutive perfect squares and, therefore, cannot itself be a perfect square. We may therefore confine our search for n which yield $n^2-11n+25$

as a perfect square to $0 \leqslant n \leqslant 11$, since the problem asks for non-negative n. However, for completeness, we may also check for $n < 0$ that

$$(n-5)^2 = n^2 - 11n + 25 + n < n^2 - 11n + 25 < n^2 - 11n + 36 - n = (n-6)^2,$$

and again $n^2 - 11n + 25$ could not be a perfect square. As before, we are left with $0 \leqslant n \leqslant 11$ for our remaining search. We might now simply test all 12 integers between and inclusive of 0 and 11, in which case our solution here would be the same strategy as in **Solution 1** (only without the simplifying transformation $n = k + 6$ used to more easily reveal the relevant inequalities). Instead of following this path, we instead exploit congruency to rule out further values of n.

If an integer $m \geqslant 0$ is even, then it is divisible by 2, implying that m^2 is divisible by 4. Therefore,[1] $m^2 \equiv 0 \pmod 4$ for even $m \geqslant 0$. If an integer $m \geqslant 1$ is odd, then $m-1$ is even, and so divisible by 2. In the identity $m^2 = (m-1)^2 + 2(m-1) + 1$ the first two terms on the right-hand side are then divisible by 4. Therefore, $m^2 \equiv 1 \pmod 4$ for odd $m \geqslant 1$. We conclude that "congruency to either 0 or 1 modulo 4" is a *necessary* condition for an integer to be a perfect square. However, such congruency is not a *sufficient* condition for an integer to be a perfect square. Indeed, $32 \equiv 0 \pmod 4$ and $29 \equiv 1 \pmod 4$, but neither 32 nor 29 is a perfect square.

These observations show that for $n^2 - 11n + 25$ to be a perfect square, we must have

$$n^2 - 11n + 25 \equiv 0 \pmod 4 \qquad \text{or} \qquad n^2 - 11n + 25 \equiv 1 \pmod 4.$$

We might analyze these expressions directly, but a simpler path is to exploit the transitive property mentioned in the footnote. Since $n^2 - 11n + 25 = n^2 + n + 1 + 4(6 - 3n)$, we see that $n^2 - 11n + 25 \equiv n^2 + n + 1 \pmod 4$. By transitivity then, for $n^2 - 11n + 25$ to be a perfect square, we must have

$$n^2 + n + 1 \equiv 0 \pmod 4 \qquad \text{or} \qquad n^2 + n + 1 \equiv 1 \pmod 4.$$

We now check for $n = 0, 1, 2, 3$ that $n^2 + n + 1 \equiv 1, 3, 3, 1 \pmod 4$. Because $n = 1, 2$ cannot work, and because $n^2 + n + 1 \equiv (n+4)^2 + (n+4) + 1 \pmod 4$, we then know that $n = 1, 2, 5, 6, 9, 10$ cannot work. Likewise, because $n^2 + n + 1 \equiv 1 \pmod 4$ for $n = 0, 3$, we also know $n^2 + n + 1 \equiv 1 \pmod 4$ for $n = 0, 3, 4, 7, 8, 11$; therefore, for these integers

[1] Given an integer $n \geqslant 1$ (the *modulus*) and two integers a and r, the notation $a \equiv r \pmod n$, read as "*a is congruent with r modulo n*", means that $a - r$ is divisible by n; in other words, $a - r = pn$ for some integer p. If $0 \leqslant r < n$, then r is the *remainder* when dividing a by n. For example, $37 \equiv 1 \pmod 4$, because $37 - 1 = 36 = 9 \times 4$; moreover, 1 is the remainder when dividing 37 by 4, because $0 \leqslant 1 < 4$. However, while $37 \equiv 9 \pmod 4$, because $37 - 9 = 28 = 7 \times 4$, now 9 is not the remainder when dividing 37 by 4, because $9 \geqslant 4$. Congruence modulo n is *transitive*: if $a \equiv b \pmod n$ (so $a - b = pn$ for some integer p) and $b \equiv c \pmod n$ (so $b - c = \ell n$ for some integer ℓ), then $a \equiv c \pmod n$ (indeed, from the assumptions $a - c = a - b + b - c = pn + \ell n = (p + \ell)n$.

$n^2 - 11n + 25$ is potentially a perfect square. So we now check $n = 0, 3, 4, 7, 8, 11$, finding

$$\begin{aligned}
0^2 - 11 \times 0 + 25 &= 5^2\\
3^2 - 11 \times 3 + 25 &= 1^2\\
4^2 - 11 \times 4 + 25 &= -3\\
7^2 - 11 \times 7 + 25 &= -3\\
8^2 - 11 \times 8 + 25 &= 1^2\\
11^2 - 11 \times 11 + 25 &= 5^2.
\end{aligned}$$

Evidently, $n = 4$ and $n = 7$ do not work, but $n = 0, 3, 8, 11$ do give rise to perfect squares. Their sum is $0 + 3 + 8 + 11 = 22$.

Solution 3. Direct observation immediately yields that when $n = 0$ then $n^2 - 11n + 25 = 25 = 5^2$, so $n = 0$ is a solution. Are there others? You may have a keen eye, and notice that when $n = 11$, then $n^2 - 11n + 25 = 11^2 - 11^2 + 25 = 25 = 5^2$, so $n = 11$ is another solution. Are there more solutions? You could try small numbers, and discover that $n = 3$ and $n = 8$ yield solutions as $3^2 - 11 \times 3 + 25 = -24 + 25 = 1 = 1^2$ and $8^2 - 11 \times 8 + 25 = 64 - 88 + 25 = 1 = 1^2$. These are the only solutions, so the sum in question is $0 + 3 + 8 + 11 = 22$. But why are these the only solutions? Could there be a very large integer n such that $n^2 - 11n + 25$ is a perfect square, and we can't simply guess it?

Let us introduce some notation. Consider $y = x^2 - 11x + 25$, the equation of an upright parabola that intersects the y-axis at $y = 25$, and intersects the x-axis at two points $x_\pm$ that you could compute using the famous formula $x_\pm = (-b \pm \sqrt{b^2 - 4ac})/(2a)$, where here $a = 1$, $b = -11$, and $c = 25$. You can verify that $(x, y) = (x_-, 0)$ and $(x, y) = (x_+, 0)$ are not integer solution pairs. We can complete the square by adding and subtracting $(\frac{11}{2})^2 = \frac{121}{4}$, observing that

$$x^2 - 11x + 25 = \left[x^2 - 2 \cdot \tfrac{11}{2}x + \left(\tfrac{11}{2}\right)^2\right] + 25 - \left(\tfrac{11}{2}\right)^2 = \left(x - \tfrac{11}{2}\right)^2 - \tfrac{21}{4}.$$

This tells us that the upright parabola is symmetric with respect to the line $x = \frac{11}{2}$. By symmetry, and since $\frac{11}{2}$ is a half integer, once we have one integer solution we can always find another integer solution (possibly negative) by reflecting across the line of symmetry. For example, $n = 3$ is a solution, and so by symmetry $n = 8$ must be a solution, since the middle point is precisely $\frac{3+8}{2} = \frac{11}{2}$. Similarly, once you have found that $n = 0$ is a solution, then $n = 11$ will be another solution, since $\frac{0+11}{2} = \frac{11}{2}$.

What we want to solve is an equation of the form $m^2 = n^2 - 11n + 25$, where n must be a non-negative integer, and m an integer. Let us solve for n in terms of m using the quadratic formula. We are then solving $n^2 - 11n + (25 - m^2) = 0$ for n, and the solutions are

$$n_\pm = \frac{11 \pm \sqrt{11^2 - 4(25 - m^2)}}{2} = \frac{11 \pm \sqrt{21 + 4m^2}}{2}. \tag{4}$$

For either n_- or n_+ to be an integer, we need the discriminant $21 + 4m^2$ to be a perfect square. In other words, we are looking for integers (k, m), such that $k^2 = 21 + 4m^2 = 21 + (2m)^2$. Reordering, we get $k^2 - (2m)^2 = 21$. We can factor the left-hand side using $a^2 - b^2 = (a - b)(a + b)$, so that

$$(k - 2m)(k + 2m) = 21.$$

Since $21 = 1 \times 21 = 3 \times 7$ are the only ways to factor 21, we will have to match the factors, $k - 2m$ and $k + 2m$, on the left-side of the last equation, with either 21 and 1, or 1 and 21, or 7 and 3, or 3 and 7. We can then analyze each case separately, and decide whether it provides a solution to the original problem or not. Here are the cases.

(i) $k - 2m = 21$ and $k + 2m = 1$. In this case, addition of both equations gives $2k = 22$; therefore, $k = 11$ and $m = -5$. Substituting $m = -5$ into (4), we find

$$n_\pm = \tfrac{1}{2}(11 \pm \sqrt{11^2}) = \tfrac{1}{2}(11 \pm 11),$$

that is, $n = n_- = 0$ or $n = n_+ = 11$.

(ii) $k - 2m = 1$ and $k + 2m = 21$. As before, addition of both equations gives $2k = 22$; therefore, $k = 11$ and this time $m = 5$. Substituting $m = 5$ into (4), we get the same solutions as in **(i)**, that is $n = 0$ or $n = 11$.

(iii) $k - 2m = 7$ and $k + 2m = 3$. As before, adding both equations, we get $2k = 10$; therefore $k = 5$ and $m = -1$. Substitution of $m = -1$ into (4) then gives

$$n_\pm = \tfrac{1}{2}(11 \pm \sqrt{21 + 4}) = \tfrac{1}{2}(11 \pm 5),$$

that is, $n = n_- = 3$ or $n = n_+ = 8$.

(iv) $k - 2m = 3$ and $k + 2m = 7$. As before, addition of both equations gives $2k = 10$; therefore, $k = 5$ and this time $m = 1$. Substituting $m = 1$ into (4), we get the same solutions as in **(iii)**, that is $n = 3$ or $n = 8$.

We have found all possible solutions, $n = 0, 3, 8, 11$. Their sum is 22.

Solution 4. Consider the factored form of the function $p(x) = x^2 - 11x + 25$, namely

$$p(x) = \Big[x - \big(\tfrac{11}{2} - \tfrac{\sqrt{21}}{2}\big)\Big]\Big[x - \big(\tfrac{11}{2} + \tfrac{\sqrt{21}}{2}\big)\Big]. \tag{5}$$

The problem is to find non-negative integers n such that $p(n)$ is a perfect square m^2. To rule out integers n which will not work, we first find the set of real numbers for which $p(x) < 0$. Toward this end, write

$$4p(x) = \underbrace{(2x - 11 + \sqrt{21})}_{\text{factor 1}}\,\underbrace{(2x - 11 - \sqrt{21})}_{\text{factor 2}}, \tag{6}$$

from which arises the following sign table.

	$2x < 11 - \sqrt{21}$	$2x = 11 - \sqrt{21}$	$11 - \sqrt{21} < 2x < 11 + \sqrt{21}$	$2x = 11 + \sqrt{21}$	$11 + \sqrt{21} < 2x$
factor 1	$\ominus$	0	$\oplus$	$\oplus$	$\oplus$
factor 2	$\ominus$	$\ominus$	$\ominus$	0	$\oplus$

The table shows that $m = 0$ does not correspond to a possible solution pair (n, m), since $p(x) = 0$ is solved by $x = \frac{1}{2}(11 \pm \sqrt{21})$, that is non-integer values of x. Now consider the inequality chain

$$\underbrace{11 - \sqrt{25}}_{6} < 11 - \sqrt{21} < \underbrace{11 - \sqrt{9}}_{8} < \underbrace{11 + \sqrt{9}}_{14} < 11 + \sqrt{21} < \underbrace{11 + \sqrt{25}}_{16}. \tag{7}$$

With this chain and the table, we see that $4p(x)$, and so also $p(x)$, is negative for $2x = \{8, 10, 12, 14\}$; whence the integers $n = \{4, 5, 6, 7\}$ cannot work. Therefore, we may confine our search to integers n for which $0 \leqslant n \leqslant 3$ (in which case both square-bracket factors are strictly negative) and $n \geqslant 8$ (in which case both square-bracket factors are strictly positive). From (5) we see that

$$p(x) = \left(x - \tfrac{11}{2}\right)^2 - \tfrac{21}{4},$$

showing that $p(x)$ is an even function about $x = \frac{11}{2}$. Therefore, we may confine our search to $n \geqslant 8$. Indeed, by symmetry

$$\begin{aligned}
p(0) &= p(\tfrac{11}{2} - \tfrac{11}{2}) = p(\tfrac{11}{2} + \tfrac{11}{2}) = p(11)\\
p(1) &= p(\tfrac{11}{2} - \tfrac{9}{2}) = p(\tfrac{11}{2} + \tfrac{9}{2}) = p(10)\\
p(2) &= p(\tfrac{11}{2} - \tfrac{7}{2}) = p(\tfrac{11}{2} + \tfrac{7}{2}) = p(9)\\
p(3) &= p(\tfrac{11}{2} - \tfrac{5}{2}) = p(\tfrac{11}{2} + \tfrac{5}{2}) = p(8).
\end{aligned}$$

Assume then that square-bracket factors in (6) *are positive.* Here are two ways to proceed.

• Similar to what was done in the first and second solutions, we might be clever enough to notice that for $n > 11$ we have

$$(n-6)^2 = n^2 - 11n + (36 - n) < n^2 - 11n + 25 < n^2 - 11n + (25 + n) = (n-5)^2,$$

in which case we could rule out all $n > 11$. Indeed, this observation shows that, for $n > 11$, the value $p(n)$ lies strictly between consecutive perfect squares. Whence $p(n)$ itself cannot be a perfect square. Our search then reduces to just four integers: $n = 8, 9, 10, 11$, a small enough set to explicitly check. Moreover, with the congruency-modulo-4 argument from **Solution 2**, we could also rule out $n = 9, 10$; leaving only $n = 8, 11$ for confirmation.

• *Let us proceed without the observations made in the last bullet-point item.* Then in (6) since $0 <$ factor 2 $<$ factor 1, it must hold that

$$(\text{factor } 2)^2 < \underbrace{(\text{factor } 2)(\text{factor } 1)}_{4p(x)} < (\text{factor } 1)^2,$$

and upon taking square roots,[2]

$$2x - 11 - \sqrt{21} < 2\sqrt{p(x)} < 2x - 11 + \sqrt{21}.$$

Therefore, with the inequalities (7) we see that a solution pair (n, m) would have to obey

$$2n - 16 < 2n - 11 - \sqrt{21} < 2m < 2n - 11 + \sqrt{21} < 2n - 6.$$

From the leftmost side of this chain, the assumption $n \geqslant 8$ (as seen above) ensures that both square-bracket factors are positive. The above chain gives $n - 8 < m < n - 3$, and the only possibilities are then $m \in \{n-7, n-6, n-5, n-4\}$. Substitution of $m = n - \alpha$ into $m^2 = n^2 - 11n + 25$ yields

$$-2\alpha n + \alpha^2 = -11n + 25,$$

from which we find the formula

$$n = \frac{\alpha^2 - 25}{2\alpha - 11}.$$

[2]The square root $\sqrt{x}$ is an increasing function which preserves order relations among positive real numbers.

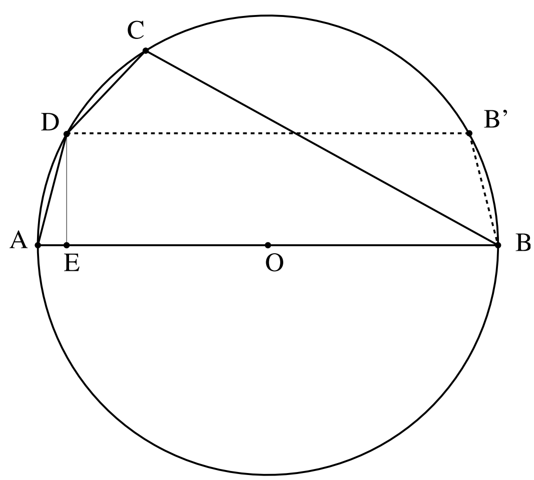
This formula yields the n-values quoted above:

$$\begin{array}{ll} n = 3 & \text{for } \alpha = 4 \\ n = 0 & \text{for } \alpha = 5 \\ n = 11 & \text{for } \alpha = 6 \\ n = 8 & \text{for } \alpha = 7. \end{array}$$

Only solutions $n = 8, 11$ follow from our analysis when both square-bracket factors are positive. However, the symmetry arguments above show $n = 0, 3$ (which a make both square-bracket factors negative) are also permissible. The sum is $0 + 3 + 8 + 11 = 22$.

9. ENTERED ANSWER: $\boxed{8}$

Quick solution 1 (due to Bill Cordwell). We exploit our knowledge that the test format requires an integer solution! Referring to the first figure, consider the right triangle ΔABC. We know $|BC| = 14$, and evidently $|CA| < 4+4 = 8$. The hypotenuse is $|AB| = 2r$, where r is the radius of the circle. The Pythagorean Theorem then gives $4r^2 = 14^2 + |CA|^2 < 14^2 + 8^2$ or $r^2 < 65$. This inequality shows that the solution r is an integer less than 9. Since $|AB| = 2r > |BC| = 14$, we have $r > 7$ necessarily. The only possible integer is $r = 8$.



Solution 2. Referring to the first figure, let $\theta = \angle AOD$. When rotated counter-clockwise by angle θ about the center O, triangle ΔBCD becomes triangle $\Delta B'DA$. Note that $|B'D| = 14$, and so $|OE| = 7$. It follows that $|AE| = r - 7$, where r is the radius of the circle. By the Pythagorean Theorem applied to the right triangle ΔAED,

$$|DE|^2 = |AD|^2 - |AE|^2 = 4^2 - (r-7)^2.$$

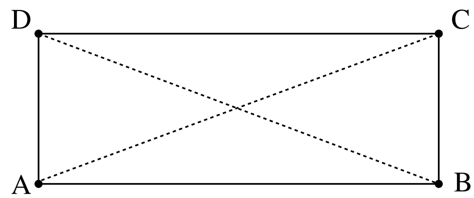
By the Pythagorean Theorem applied to the right triangle ΔDEO,

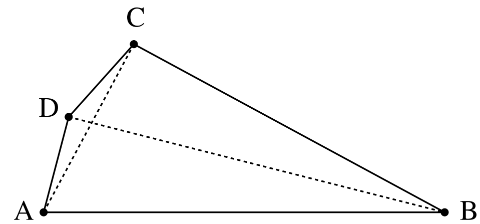
$$|DE|^2 = r^2 - 7^2.$$

Equating these to expressions, we find $4^2 - (r-7)^2 = r^2 - 7^2$ and so

$$2r^2 - 14r - 16 = (2r - 16)(r + 1) = 0.$$

Since the radius must be positive, $r = 8$.





Solution 3 (expanded account of the argument made by Sean Choi). Referring to the first figure, let r be the radius of the circle, so that $|AB| = 2r$. Both $\triangle ABC$ and $\triangle ABD$ are inscribed in the circle, and each has the diameter of the circle as one leg. We then know that both $\triangle ABC$ and $\triangle ABD$ are right triangles; more precisely, $\angle BDA$ and $\angle BCA$ are right angles[3]. Invocation of the *Pythagorean Theorem* for $\triangle ABC$ gives

$$|AC|^2 = |AB|^2 - |BC|^2 = (2r)^2 - 14^2 = 4(r^2 - 49), \tag{8}$$

and likewise invocation of the *Pythagorean Theorem* for $\triangle ABD$ gives

$$|BD|^2 = |AB|^2 - |AD|^2 = (2r)^2 - 4^2 = 4(r^2 - 4). \tag{9}$$

Although it may not be readily apparent, $\square ABCD$ is a quadrilateral, and a *cyclic* one since it inscribes the circle. We may then appeal to *Ptolemy's Theorem:*

> *The product of (the lengths of) the diagonals of a cyclic quadrilateral equals the sum of the products of (the lengths of) its opposite sides.*

One most likely pictures the theorem with the rectangle shown at the top of the second figure, in which case $|AC| \cdot |BD| = |AD| \cdot |BC| + |AB| \cdot |CD|$ (why?). Note that the diagonals of this rectangle would be diameters of the circumscribing circle (not shown). However, the result holds just as well for the kite-shaped quadrilateral shown at the bottom of the second figure. This is the cyclic quadrilateral from the first figure. The diagonal lengths of the kite-shaped figure have already been computed in (8) and (9). Therefore, invocation of *Ptolemy's Theorem* yields the following statement.

$$\overbrace{\sqrt{16(r^2-49)(r^2-4)}}^{|AC|\cdot|BD|} = \overbrace{4 \cdot 14}^{|AD|\cdot|BC|} + \overbrace{8r}^{|AB|\cdot|CD|}$$

Simplification of this equation gives

$$\sqrt{(r^2-49)(r^2-4)} = 14 + 2r,$$

[3]Consider the angle formed by 3 points on a circle, and the arc of the circle subtending this angle. Recall that this angle is always half the center angle subtended by the same arc (why?). For example, in the first figure, the angle $\angle ACB$ is subtended by the arc $\widehat{\mathrm{AB}}$ (running counterclockwise from A to B), and the center angle subtended by the same arc $\widehat{\mathrm{AB}}$ would be $\angle AOB = 180°$, therefore $\angle ACB = \angle AOB/2 = 180°/2 = 90°$.

and upon squaring both sides

$$(r^2 - 49)(r^2 - 4) = 4(r + 7)^2.$$

Further algebra then gives

$$\begin{aligned} 0 &= (r^2 - 49)(r^2 - 4) - 4(r + 7)^2 \\ &= (r + 7)[(r - 7)(r^2 - 4) - 4(r + 7)] \\ &= (r + 7)(r^3 - 7r^2 - 8r) \\ &= r(r + 7)(r + 1)(r - 8). \end{aligned}$$

The only strictly positive solution is $r = 8$.

10. ENTERED ANSWER: $\boxed{10}$

Solution. We form

$$u^2 = \tfrac{1}{4}x^2 r^{-2}(r + r^{-1})^2, \qquad v^2 = \tfrac{1}{4}y^2 r^{-2}(r - r^{-1})^2,$$

showing that

$$\frac{4u^2}{(r + r^{-1})^2} + \frac{4v^2}{(r - r^{-1})^2} = x^2 r^{-2} + y^2 r^{-2} = 1.$$

This is the equation of an ellipse in the (u, v)-plane with semi-major axis $a = \frac{1}{2}(r + r^{-1})$ and semi-minor axis $b = \frac{1}{2}|r - r^{-1}|$. Here we demand that $r + r^{-1} = \frac{10}{3}$ and $|r - r^{-1}| = \frac{8}{3}$. The first equation is solved by $r = \frac{1}{3}$ and $r = 3$. We then confirm for $r = 3$ that $r - r^{-1} = \frac{8}{3}$, and for $r = \frac{1}{3}$ that $r - r^{-1} = -\frac{8}{3}$, so that the second equation $|r - r^{-1}| = \frac{8}{3}$ is also obeyed in both cases. Therefore, $r = 3$ and $r = \frac{1}{3}$ are possible radii. The reported answer is $3(3 + \frac{1}{3}) = 10$.

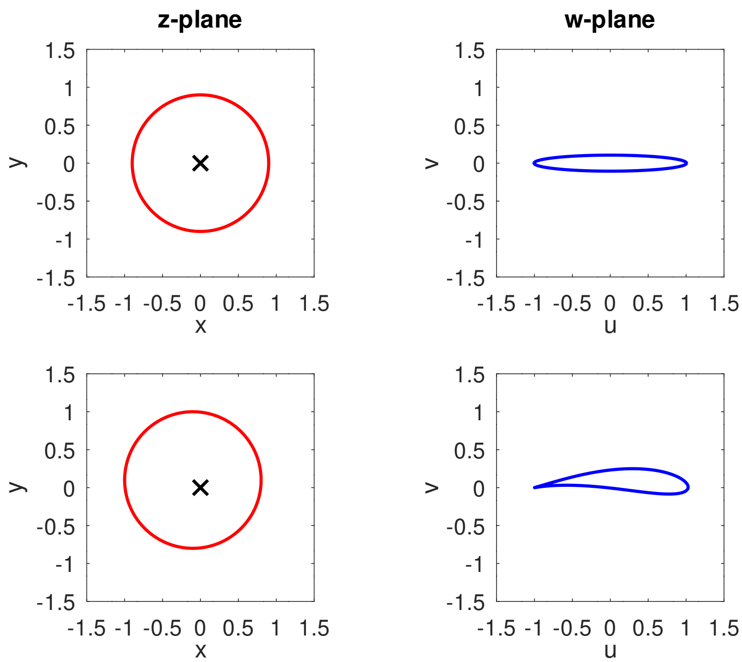
Afterward. If you know something about complex numbers, you might think to form

$$\begin{aligned} u + \mathrm{i}v &= \tfrac{1}{2}x(1 + r^{-2}) + \tfrac{1}{2}\mathrm{i}y(1 - r^{-2}) \\ &= \tfrac{1}{2}(x + \mathrm{i}y) + \tfrac{1}{2}(x - \mathrm{i}y)r^{-2}. \end{aligned}$$

Then with $w = u + \mathrm{i}v$ and $z = x + \mathrm{i}y$, we have, upon using $\bar{z} = x - \mathrm{i}y$ and $r^2 = z\bar{z}$, that

$$w = \tfrac{1}{2}\big(z + 1/z\big).$$

This is Joukowski's transformation which in the early 20th century served as a theoretical model of a wing profile (an airplane wing cross-section). It was superseded by more refined models (some also involving complex analysis), and eventually computational (i.e. computer) models. In the figure the circle of radius $r = \frac{9}{10}$ centered at the origin (top-left panel) is mapped to a thin ellipse with semimajor axis $a = \frac{181}{180}$ and semiminor axis $b = \frac{19}{180}$ (top-right panel). When the circle is shifted off center by $\frac{1}{10}(-1 + \mathrm{i})$ (bottom-left panel) the map yields the closed curved which looks like the cross-section of a wing (bottom-right panel).


z-plane
w-plane
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5 -1 -0.5 0 0.5 1 1.5
x
y
u
v

UNM-PNM Round 1

Solutions

Sean Choi

November 2024

Solution 1. Note that $\frac{2023}{2024} = 1 - \frac{1}{2024}$ and that $\frac{2024}{2025} = 1 - \frac{1}{2025}$. Since $\frac{1}{2024} > \frac{1}{2025}$, we have

$$\frac{2023}{2024} = 1 - \frac{1}{2024} < 1 - \frac{1}{2025} = \frac{2024}{2025}.$$

Thus, our answer is $\boxed{2025}$.

Solution 2. By inspection note that 2 is the only valid units digit of the smaller number. The digits $4, 5, 7$ will not work since the leftmost digit of the multiple will not be one of $2, 4, 5$, or 7 or the multiple will be a 5 digit number.

Now, since the largest possible number is 7542 and the smallest possible number is 2457, we only need to consider multiplication by 2 or 3. Otherwise the multiple will clearly exceed 7542.

Next, we consider the last digit. If we are multiplying by 2, the last digit can go from 5 to 0, 7 to 4, or from 4 to 8. From these possibilities, only going to 7 to 4 satisfies the conditions given in the problem. However, it is easy to check that neither 2457 nor 2547 work.

When we multiply by 3, the units digit can go from 5 to 5, 7 to 1, or from 4 to 2. The only cases that work are when the units digit goes from 5 to 5 and from 4 to 2. Then, we can check the possibilities:

$$2475, 2745, 2574, 2754.$$

From these, note that $2475 \cdot 3 = 7425$, which is what we are looking for. Thus, our answer is $2475 + 7425 = \boxed{9900}$.

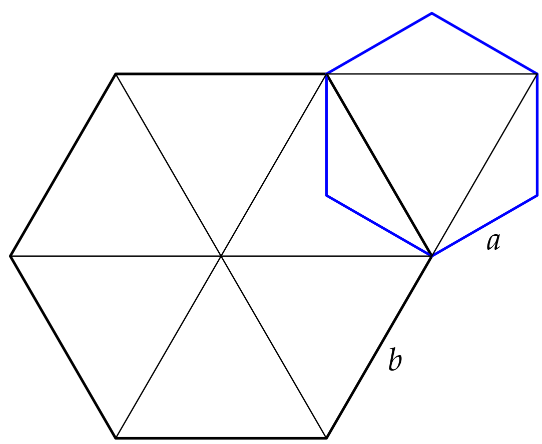
Solution 3. The optimal strategy for Cora is to force the game from a 3-pile game to a 2-pile game. This means that Cora should take all the stones from the first pile.

Then, if Ernie is playing optimally as well, Ernie must take some n stones ($n < 7$) from one of the two piles remaining. Otherwise (i.e., if $n = 7$), the game will become a one-pile game from which Cora will be able to win by taking all the stones.

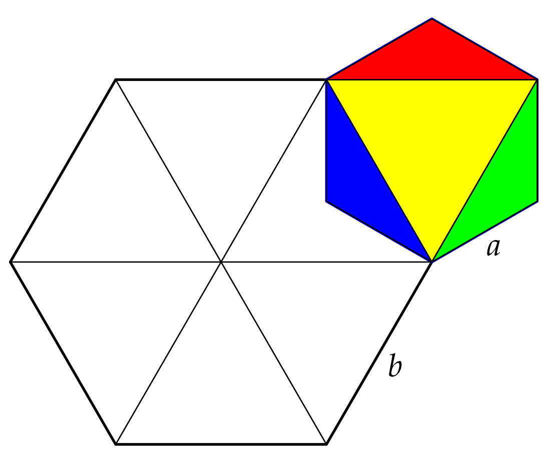
Then, Cora can take n but from the other pile, effectively "mirroring" Ernie's moves. If Cora plays in this manner, Cora will be able to reduce the game to a situation where the second player has two single-stone piles left. This means that they must take one stone from a pile, leaving Cora to win the game by taking the last stone.

So, Cora will always be able to win when playing optimally. Thus, the answer is $\boxed{1}$.

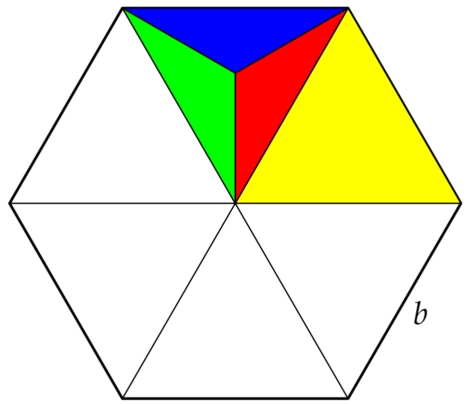
Solution 4. A visual solution. First, we decompose the hexagons as follows:



In order to clearly see what is going on, we will add a bit of color:



Rearranging the areas, we have:

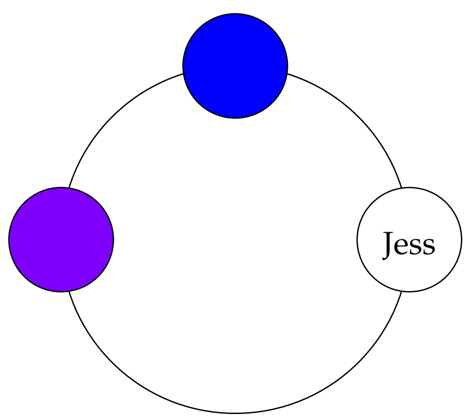


Thus, the ratio of the areas is $\frac{6}{2} = \boxed{3}$.

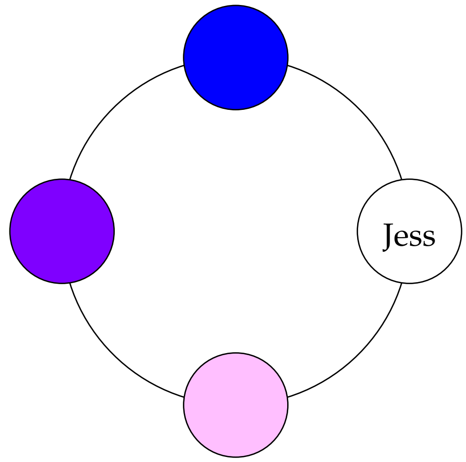
Solution 5. Note that $x = \sqrt{1 + \sqrt{1 + \sqrt{1}}} = \sqrt{1 + \sqrt{2}}$. Now,

$$\begin{aligned} x^4 - 2\sqrt{2} = (x^2)^2 - 2\sqrt{2} &= (1 + \sqrt{2})^2 - 2\sqrt{2} \\ &= (1 + 2\sqrt{2} + 2) - 2\sqrt{2} \\ &= \boxed{3}. \end{aligned}$$

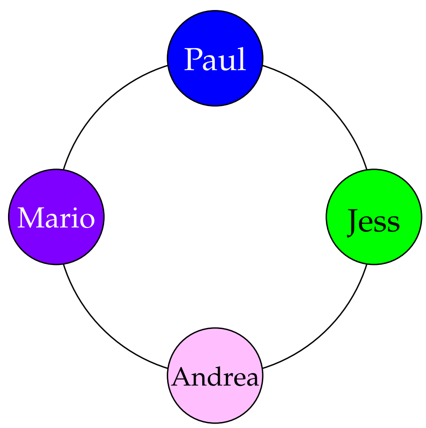
Solution 6. Given the first condition, we can let the person with purple hair, the person with blue hair, and Jess be 3 people in order on the circle:



Now, we consider who can have pink hair. If Jess has Pink hair, then Mario must have blue hair, a contradiction. Thus, the person with pink hair must take the last remaining spot in the circle:



This means that Jess must have green hair and that Mario has purple hair. Since Andrea cannot have blue hair, Paul must have blue hair and Andrea must have pink hair:



Filling in the table according to the numbers corresponding to the hair colors given, we find that the answer is $\boxed{4321}$.

Solution 7. Let

$$\begin{aligned}
a &= \text{lifespan of bowhead whale}\\
b &= \text{lifespan of brown bear}\\
c &= \text{lifespan of western gorilla}\\
d &= \text{lifespan of wolverine}\\
e &= \text{lifespan of wild turkey}\\
f &= \text{lifespan of tiger}\\
g &= \text{lifespan of elephant.}
\end{aligned}$$

From the information given in the problem, we construct the system of equations

$$\begin{cases}
a &= 8b\\
b &= c - 11\\
c &= 3d\\
d &= e + 8\\
e &= f - 10\\
f &= \frac{1}{4}g\\
g &= \dfrac{a+b+c+d+e+f+g-11}{6}.
\end{cases}$$

Starting from the fourth equation, we can work upwards and write a, b, c, and d in terms of e. We can also write the 5th and 6th equations in terms of e, which is what we wish to find. Doing so yields

$$\begin{cases}
a &= 24e + 104\\
b &= 3e + 13\\
c &= 3e + 24\\
d &= e + 8\\
f &= e + 10\\
g &= 4e + 40.
\end{cases}$$

Then, we can substitute into

$$g = \frac{a+b+c+d+e+f+g-11}{6}$$

and simplify to get

$$4e + 40 = \frac{1}{6}(37e + 188).$$

This gives

$$24e + 240 = 37e + 188$$

which yields the solution $e = \boxed{4}$ for the lifespan of a wild turkey.

Solution 8. Since n is nonnegative,

$$n^2 - 11n + 25 < n^2 - 10n + 25 = (n-5)^2.$$

Now, note that the closest perfect square below $(n-5)^2$ is

$$(n-6)^2 = ((n-5)-1)^2 = (n-5)^2 - 2(n-5) + 1.$$

Then, if

$$(n-6)^2 < n^2 - 11n + 25 < (n-5)^2$$

is satisfied, then $n^2 - 11n + 25$ will always be between two consecutive perfect squares, which means that it can never be a perfect square. Simplifying the inequality, we get

$$\begin{aligned} n^2 - 12n + 36 < n^2 - 11n + 25 < n^2 - 10n + 25 \\ -12n + 36 < -11n + 25 < -10n + 25 \end{aligned}$$

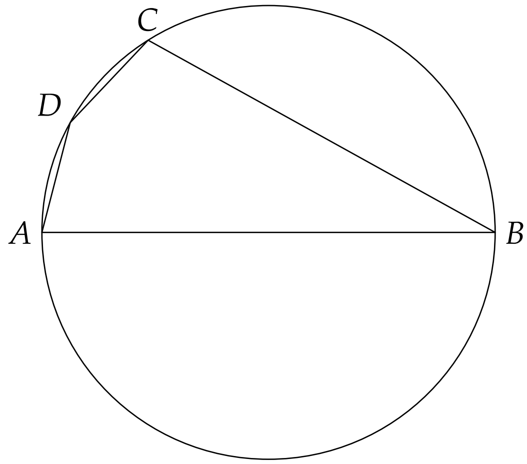
So $n > 11$ is when this is satisfied. Thus, it suffices to test $n \leq 11$.

Now, we take the expression modulo 4 in order to narrow down the cases we test. We have

$$n^2 - 11n + 25 \equiv n^2 + n + 1 \pmod{4}.$$

Testing residues modulo 4, the only two residues that work are $n \equiv 0 \pmod{4}$ and $n \equiv 3 \pmod{4}$, which means that we only have to test $n = 0, 4, 8, 3, 7, 11$. We find that $n = 4, n = 7$ do not yield perfect squares so our answer is $0 + 8 + 3 + 11 = \boxed{22}$.

Solution 9. Let the radius be r. Then $AB = 2r$. Since $\overline{AB}$ is the diameter of the circle, $\triangle ABD$ and $\triangle ABC$ are right. Thus, applying the Pythagorean Theorem, we have that $AC = \sqrt{(2r)^2 - 14^2}$ and $BD = \sqrt{(2r)^2 - 4^2}$.



Since $ABCD$ is a cyclic quadrilateral, we can apply Ptolemy's theorem, which states that $AC \cdot BD = AB \cdot CD + BC \cdot AD$. So,

$$\sqrt{(2r)^2 - 14^2} \cdot \sqrt{(2r)^2 - 4^2} = (2r) \cdot 4 + 14 \cdot 4.$$

Simplifying, we have

$$\sqrt{(2r)^2 - 14^2} \cdot \sqrt{(2r)^2 - 4^2} = 8r + 56,$$

$$\sqrt{r^2 - 49} \cdot \sqrt{r^2 - 4} = 2r + 14.$$

Squaring both sides,

$$\begin{aligned}
(r^2 - 49)(r^2 - 4) &= (2r + 14)^2, \\
(r + 7)(r - 7)(r^2 - 4) &= 4(r + 7)^2, \\
(r + 7)(r^3 - 7r^2 - 4r + 28) &= (r + 7)(4r + 28), \\
(r + 7)(r^3 - 7r^2 - 8r) &= 0, \\
r(r + 7)(r^2 - 7r - 8) &= 0.
\end{aligned}$$

Factoring,

$$r(r + 7)(r - 8)(r + 1) = 0.$$

We take the positive solution, so $r = \boxed{8}$.

Solution 10. We have

$$u^2 = x^2 \left(\frac{1 + r^{-2}}{2} \right)^2$$

and

$$v^2 = y^2 \left(\frac{1 - r^{-2}}{2} \right)^2.$$

Then,

$$x^2 = u^2 \left(\frac{2}{1 + r^{-2}} \right)^2$$

and

$$y^2 = v^2 \left(\frac{2}{1 - r^{-2}} \right)^2.$$

Then, we map the equation for the circle

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

to

$$\frac{u^2 \left(\frac{2}{1+r^{-2}} \right)^2}{r^2} + \frac{v^2 \left(\frac{2}{1-r^{-2}} \right)^2}{r^2} = 1.$$

Then, note that

$$\left(\frac{2}{1+r^{-2}} \right)^2 \cdot \frac{1}{r^2} = \frac{1}{a^2}$$

and

$$\left(\frac{2}{1-r^{-2}} \right)^2 \cdot \frac{1}{r^2} = \frac{1}{b^2}.$$

Thus, since $a = \dfrac{5}{3}$ and $b = \dfrac{4}{3}$, we have

$$\left| \frac{2}{r + r^{-1}} \right| = \frac{3}{5}$$

and

$$\left| \frac{2}{r - r^{-1}} \right| = \frac{3}{4}.$$

Then,

$$r + r^{-1} = \frac{10}{3}$$

and

$$|r - r^{-1}| = \frac{8}{3}.$$

From the first equation, we get $r = \frac{1}{3}$ and $r = 3$. We can confirm that both these values satisfy the second equation as well. Thus, the desired sum is $\dfrac{10}{3}$ and so the answer is $\boxed{10}$.