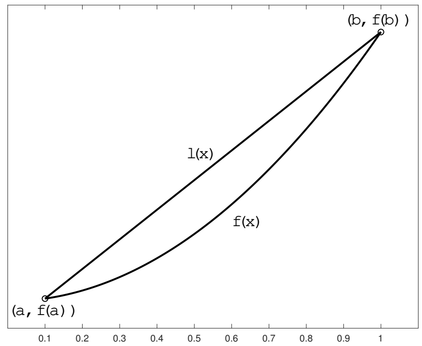
UNM – PNM STATEWIDE MATHEMATICS CONTEST LII

February 1, 2020 Second Round Three Hours

1. Calculate the sum of the roots for $|2x - 4| = 5$.

2. Show that there exist two numbers, among any $k + 1$ numbers, such that their difference is divisible by k.

3. For any $p > 1$ and $a, b > 0$, we have the property that the graph of the function $f(x) = x^p$ for $a \leq x \leq b$ lies below the line segment joining $(a, f(a)\ )$ and $(b, f(b))$. That is, let $(x, l(x))$ denote the points on the line segment joining $(a, f(a))$ and $(b, f(b))$. Then we have $f(x) < l(x)$ for $a < x < b$, as illustrated in the figure below for $a = 0.1, b = 1$ and $p = 2$.



Use this fact to show that

$$(2020)^{2020} < 2^{2019}(2000^{2020} + 20^{2020})$$

4. Let us consider a curve C and a line l in $\mathbb{R}^2$. The equation of the curve C is $y = \sqrt{-x^2 - 2x}$ and the equation of the line l is $x + y - m = 0$. For a particular value of m, l and C may either not intersect, or interact at one point, or intersect at two different points. Determine all the possible m such that C and l have two different intersection points.

5. Let

$$x, a_1, a_2, a_3, y \qquad \text{and} \qquad b_1, x, b_2, 2y, b_3$$

be the terms from two geometric sequences, where $x \neq 0$ and $y \neq 0$. Calculate $\dfrac{b_3 a_1^8}{a_2^8 b_1}$.

6. Let $x = 0.82^{0.5}, y = \sin(1), z = \log_3(\sqrt{7})$. Determine the largest and smallest numbers among x, y, z.

7. Let $\triangle ABC$ be an acute triangle. $\angle C = 2\angle A$ and $2|AC| = |AB| + |BC|$. Calculate $\sin \angle A$.

8. Let x_1, x_2, x_3 be the three roots of $x^3 - x + 1 = 0$. Calculate $x_1^5 + x_2^5 + x_3^5$.

9. What is the remainder when the number

$$12+11(13^1)+10(13^2)+9(13^3)+8(13^4)+7(13^5)+6(13^6)+5(13^7)+4(13^8)+3(13^9)+2(13^{10})+13^{11}$$

is divided by 6?

10. A five-digit number is formed by randomly choosing (with no repetitions) a number from $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ for each digit. For example, 10234 is a valid number, while 12325 is not (since 2 is repeated in the latter number).
    (a) Show that the number thus formed is divisible by 9 if and only if the sum of its digits is divisible by 9 (e.g. $92034 \bmod 9 = 0$ and $(9 + 2 + 0 + 3 + 4) \bmod 9 = 18 \bmod 9 = 0$).
    (b) What is the probability that the five-digit number thus formed is divisible by 90?