UNM - PNM STATEWIDE MATHEMATICS CONTEST L

February 3, 2018 Second Round Three Hours

1. Let $x \neq y$ be two real numbers. Let x, a_1, a_2, a_3, y and b_1, x, b_2, b_3, y, b_4 be two arithmetic sequences. Calculate $\frac{b_4-b_3}{a_2-a_1}$.

   Answer: $\frac{8}{3}$.

   Solution: Notice that $a_2 - a_1 = \frac{1}{4}(y-x),\ b_4 - b_3 = \frac{2}{3}(y-x)$. So $\frac{b_4-b_3}{a_2-a_1} = \frac{8}{3}$.

2. Determine all positive integers a such that $a < 100$ and $a^3 + 23$ is divisible by 24.

   Answer: 1, 25, 49, 73, 97.

   Solution: By assumption we have $24 \mid a^3 - 1$. It is easy to see that $a \equiv 1 \pmod 8$ and $a \equiv 1 \pmod 3$. Thus $a \equiv 1 \pmod{24}$. Since $a < 100$, we conclude that $a = 1, 25, 49, 73, 97$.

3. Let $a_1 < a_2 < a_3$ be three positive integers in the interval $[1, 14]$ satisfying $a_2 - a_1 \geq 3$ and $a_3 - a_2 \geq 3$. How many different choices of (a_1, a_2, a_3) exist?

   Answer: 120.

   Solution: Let $a_1' = a_1,\ a_2' = a_2 - 2,\ a_3' = a_3 - 4$. Then $1 \leq a_1' < a_2' < a_3' \leq 10$. So we have $\binom{10}{3} = 120$ different choices.

4. Suppose $ABCD$ is a parallelogram with area $39\sqrt{95}$ square units and $\angle DAC$ is a right angle. If the lengths of all the sides of $ABCD$ are integers, what is the perimeter of $ABCD$?

   Answer: 90 units. Suppose the length of DC is y and the length of AD is x and the length of AC is z. By the Pythagorean theorem $x^2 + z^2 = y^2$. Also since $\angle DAC$ is a right angle the area of $ABCD$ is $xz = 39\sqrt{65}$. Multiplying the Pythagorean relation by x^2 on both sides we obtain $x^4 + x^2z^2 = x^2y^2 = x^4 + (39)^2 \cdot 95 = x^2y^2$. Thus $x^2(y^2 - x^2) = (39)^2 \cdot 95$. Since x and y are integers x must divide 39. So $x = 1, 3, 13$ or 39. Testing these yields $x = 13$ and $y = 32$ implying the perimeter is 90.

5. Let x and y be two real numbers satisfying $x - 4\sqrt{y} = 2\sqrt{x-y}$. What are all the possible values of x?

   Answer: $\{0\} \cup [4, 20]$.

   Solution: Let $a = \sqrt{y},\ b = \sqrt{x-y},\ (a, b \geq 0)$. Then $x = a^2 + b^2$. The assumption implies that
$$a^2 + b^2 - 4a = 2b.$$
   By completing the squares, we get
$$(a-2)^2 + (b-1)^2 = 5, \quad a, b \geq 0.$$
   Let C be the circle with center $= (2, 1)$ and radius $= \sqrt{5}$. Let $\tilde{C}$ be the intersection of C and the first quadrant. Then for $a, b \in \tilde{C}$, we either have $a = b = 0$ or $a^2 + b^2 \in [4, 20]$.

6. A round robin chess tournament took place between 16 players. In such a tournament, each player plays each of the other players exactly once. A win results in a score of 1 for the player, a loss results in a score of -1 for the player and a tie results in a score of 0. If at least 75 percent of the games result in a tie, show that at least two of the players have the same score at the end of the tournament.

   Answer: There are $\binom{16}{2} = 15 \cdot 8 = 120$ matches between the players and at least $\lceil .75 \cdot 120 \rceil = \lceil 90 \rceil = 90$ were a tie, then there were at most $120 - 90 = 30$ matches that resulted in wins or losses. Suppose the 16 players had 16 different scores. This means there was at most one player with a zero. We proceed by the pigeonhole principle. The two "pigeonholes" are the set of negative numbers and the set of positive numbers. Since there are at least 15 players with non zero scores, at least $\lceil 15/2 \rceil = 8$ players had either positive scores or negative scores. Since the players supposedly had all different scores then the absolute value of the sum of the scores is at least $1+2+3+4+5+6+7+8 = \frac{8 \cdot 9}{2} = 36$ implying that at least 36 games resulted in a win or loss, contradicting the fact that at most 30 matches resulted in a win or loss.

7. Let a, b be positive real numbers such that $\frac{1}{a} + \frac{1}{b} = 1$. Show that
$$(a+b)^{2018} - a^{2018} - b^{2018} \geq 2^{2 \cdot 2018} - 2^{2019}.$$

Solution: We are going to show that for any positive integer n,
$$(a+b)^n - a^n - b^n \geq 2^{2n} - 2^{n+1}.$$
By assumption we have $ab = a + b$. And $a + b \geq 2\sqrt{ab}$. So we have $a + b = ab \geq 4$. Thus $(a+b)^n = (ab)^n \geq 2^{2n}$. Also we have $a^n + b^n \geq 2\sqrt{(ab)^n} \geq 2^{n+1}$.

Now let us finish the rest of the proof by induction.

Step 1. Let $n = 1$. The inequality holds.

Step 2. Suppose the inequality holds for $n = k$. Then for $n = k + 1$, we have
$$\begin{aligned}
&(a+b)^{k+1} - a^{k+1} - b^{k+1} \\
=&(a+b)(a+b)^k - (a+b)(a^k + b^k) + ab(a^{k-1} + b^{k-1}) \\
=&(a+b)((a+b)^k - a^k - b^k) + ab(a^{k-1} + b^{k-1}) \\
\geq&4(2^{2k} - 2^{k+1}) + 4 \cdot 2^k \\
\geq&2^{2(k+1)} - 2^{k+2}.
\end{aligned}$$

Thus the inequality holds for all n and in particular for $n = 2018$.

8. Using red, blue and yellow colored toothpicks and marshmallows, how many ways are there to construct distinctly colored regular hexagons? (Note that two colored hexagons are the same if we can either rotate one of the hexagons and obtain the other or flip one of the hexagons about some line and obtain the other.)

Answer: 92

We can think of the hexagons as the disjoint union of the sets:

$$\begin{aligned}
X_1 &= \{\text{all sides have the same color}\} \\
X_{2,(1,5)} &= \{\text{2 distinct colors distributed in the ratio 1:5}\} \\
X_{2,(2,4)} &= \{\text{2 distinct colors distributed in the ratio 2:4}\} \\
X_{2,(3,3)} &= \{\text{2 distinct colors distributed in the ratio 3:3}\} \\
X_{3,(1,1,4)} &= \{\text{3 distinct colors distributed in the ratio 1:1:4}\} \\
X_{3,(1,2,3)} &= \{\text{3 distinct colors distributed in the ratio 1:2:3}\} \\
X_{3,(2,2,2)} &= \{\text{3 distinct colors distributed in the ration 2:2:2}\}
\end{aligned}$$

Since these sets are disjoint, if we determine the cardinality of each of these sets, then the sum of these sizes will be the number of distinct colorings.

Since there are 3 colors, $|X_1| = 3$. To determine the size of $X_{2,(1,5)}$, note that there are $\binom{3}{2}$ ways to choose the 2 colors and $\binom{2}{1}$ ways to decide which of the two colors appears 5 times. Since any rotation or mirror symmetry will be equivalent to the coloring $xyyyyy$ where x and y are the 2 colors, we see that $|X_{2,(1,5)}| = 3 \cdot 2 = 6$.

Determining the size of $X_{2,(2,4)}$ is not only a matter of choosing the colors, but also the placements. The placements preserved by symmetry are $xxyyyy$, $xyxyyy$, $xyyxyy$, thus $|X_{2,(2,4)}| = 3 \cdot 2 \cdot 3 = 18$.

Similarly, we have to consider placement when determining the size of $X_{2,(3,3)}$. The placements preserved by symmetry are $xxxyyy$, $xxyxyy$, $xyxyxy$, thus $|X_{2,(3,3)}| = 3 \cdot 3 = 9$.

In determining the size of $X_{3,(1,1,4)}$, the colors are all used so it is a matter of choosing which color occupies 4 sides and the possible placement of the 4 sides with the same color. Because of reflections and rotations the ordering of the two singletons does not matter. The possible placements are $xxxxyz$, $xxxyxz$, $xxyxxz$. Hence, $|X_{3,(1,1,4)}| = 3 \cdot 3 = 9$.

In determining the size of $X_{3,(1,2,3)}$, the colors are all used so it is a matter of choosing how to distribute the colors among the three different numbers of colors of sides and to determine the disinct placements upto symmetry. The possible placements are $xxxyyz$, $xxxyzy$, $xxyyxz$, $xxyzxy$, $xxzyxy$, $xyxyxz$ Hence, $|X_{3,(1,2,3)}| = 3 \cdot 2 \cdot 6 = 36$.

In determining the size of $X_{3,(2,2,2)}$, the colors are all used so it is a matter of choosing the disinct placements upto symmetry. The possible placements are $xxyyzz$, $xyzxyz$, $xxyzyz$, $xxyzzy$, $xyxzyz$. Hence, $|X_{3,(1,2,3)}| = 1 + 1 + 9 = 11$. Addding these up we obtain 92 colored hexagons.

For those that know Burnside's formula the group D_6 of all symmetries of the regular hexagon acting on the set X of all labellings of hexagons which has 3^6 elements. Let n be the number of distinctly labelled hexagons. Then $n \cdot |D_6| = \sum_{g \in D_6} |X_g|$ where X_g is the elements of X fixed by the group element g. Let r be the rotation of 60 degrees and r^n denote rotation by $60n$ degrees. Let s_1, s_2 and s_3 be the reflections about a pair of opposite midpoints and t_1, t_2 and t_3 be reflections about the largest diagonals. The sizes of the sets $|X_{r^0}| = 6^3$ and $|X_r| = |X_{r^5}| = 3$ since the only hexagons fixed by rotations of 60 or 300 degrees are when the sides are all the same. $|X_{r^2}| = |X_{r^4}| = 9$ since rotations of 120 and 240 degrees only preserve the colors of every other side. $|X_{r^3} = 27|$ since rotating by 180 degrees allows us three degrees of freedom. Sides 1 and 4 will be the same, 2 and 5 will be the same and 3 and 6 will be the same. $|X_{s_i}| = 3^4$ and $|X_{t_i}| = 3^3$. Thus $n \cdot 12 = 3^6 + 2 \cdot 3 + 2 \cdot 9 + 4 \cdot 27 + 3 \cdot 81$ or $n = 92$.

9. Find the number of 4-tuples (a, b, c, d) with a, b, c and d positive integers, such that $x^2 - ax + b = 0$, $x^2 - bx + c = 0$, $x^2 - cx + d = 0$ and $x^2 - dx + a = 0$ have integer roots.

   Answer: 11

   Let x_1 and x_2 be the roots of $x^2 - ax + b = 0$. Let x_3 and x_4 be the roots of $x^2 - bx + c = 0$. Let x_5 and x_6 be the roots of $x^2 - cx + d = 0$. Let x_7 and x_8 be the roots of $x^2 - dx + a = 0$. We obtain

$$
\begin{aligned}
a &= x_1 + x_2 = x_7 x_8, \\
b &= x_3 + x_4 = x_1 x_2, \\
c &= x_5 + x_6 = x_3 x_4, \\
d &= x_7 + x_8 = x_5 x_6.
\end{aligned}
$$

   Adding the equations together we obtain

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = x_1 x_2 + x_3 x_4 + x_5 x_6 + x_7 x_8.$$

   This is equivalent to

$$(x_1 - 1)(x_2 - 1) + (x_3 - 1)(x_4 - 1) + (x_5 - 1)(x_6 - 1) + (x_7 - 1)(x_8 - 1) = 4.$$

   The summands on the left are all nonnegative integers corresponding to the partitions of 4: $(0, 0, 0, 4), (0, 0, 1, 3), (0, 0, 2, 2), (0, 1, 1, 2), (1, 1, 1, 1)$. We will check cases to determine the solutions. We will assume that $x_1 \leq x_2$, $x_3 \leq x_4$, $x_5 \leq x_6$ and $x_7 \leq x_8$.

If one of these summands is 4, say $(x_1 - 1)(x_2 - x_1)$, then a) $x_1 = 2$ and $x_2 = 5$ or b) $x_1 = 3$ and $x_2 = 3$. In case a), using that $x_1x_2 = 10 = x_3 + x_4$ and the summand involving x_3 and x_4 is 0, then $x_3 = 1$ and $x_4 = 9$. Now $x_3x_4 = 9 = x_5 + x_6$ and the summand involving x_5 and x_6 is also 0 so $x_5 = 1$ and $x_6 = 8$. Further $x_5x_6 = 8 = x_7 + x_8$ and the summand involving x_7 and x_8 is 0 so $x_7 = 1$ and $x_8 = 7$. Note that $2 + 5 = 1 \cdot 7$. In this case we obtain $a = 7$, $b = 10$, $c = 9$ and $d = 8$. We could have assumed that the summand involving x_3 and x_4 was 4, this would give us $a = 8$, $b = 7$, $c = 10$ and $d = 9$ or the summand involving x_5 and x_6 was 4, this would give us $a = 9$, $b = 8$, $c = 7$ and $d = 10$ or the summand involving x_7 and x_8 was 4, this would give us $a = 10$, $b = 9$, $c = 8$ and $d = 7$. (4 total)

In case b) we would have $x_1x_2 = 9 = x_3 + x_4$ and as above we would obtain $x_3 = 1$ and $x_4 = 8$ which implies $x_3x_4 = 8 = x_5 + x_6$ or $x_5 = 1$ and $x_6 = 7$ and $x_5x_6 = 7 = x_7 + x_8$ or $x_7 = 1$ and $x_8 = 6$. Note that $x_7x_8 = 6 = 3 + 3 = x_1 + x_2$, implying that $a = 6$, $b = 9$, $c = 8$ and $d = 7$. Permuting like above we also have $a = 7$, $b = 6$, $c = 9$, $d = 8$, or $a = 8$, $b = 7$, $c = 6$, $d = 9$ or $a = 9$, $b = 8$, $c = 7$, $c = 6$. (4 total)

If one of the summands is 3 and one is 1 or one of the summands is 2 and two are 1, arguing as above we will obtain a contradiction so we obtain no 4 tuples in these cases.

If two are the summands are 2, as above we obtain $x_1 = 2$ and $x_2 = 3$. Then $x_3 + x_4 = 6$. and $x_3 = 1$ and $x_4 = 5$ will be the only case when the summand is 2 or 0. Then we will obtain $5 = x_5 + x_6 = 2 + 3$ otherwise we get a contradiction and $6 = x_7 + x_8 = 1 + 5$. So in this case we get the 4-tuples $(5, 6, 5, 6)$ and $(6, 5, 6, 5)$. (2 total)

If all four of the summands are 1, we obtain $x_i = 2$ for all i and $a = b = c = d = 4$. (1 total). We now see that there were a total of 11 4-tuples that will give integer solutions to this set of 4 quadratic equations.

10. Let A, B, C and D be points in the Cartesian plane each a distance 1 from the origin $(0, 0)$. We define addition of points in the plane componentwise (If $P = (p_x, p_y)$ and $Q = (q_x, q_y)$, then $P + Q = (p_x + q_x, p_y + q_y)$). Show $A + B + C + D = (0, 0)$ if and only if A, B, C and D are the vertices of a rectangle.

Answer: We will denote the distance of a point $P = (p_x, p_y)$ to the origin by $|P| = \sqrt{p_x^2 + p_y^2}$. If $A + B + C + D = (0, 0)$, then $A + B = -(C + D)$. Thus $|A + B| = | - (C + D)| = |C + D|$. Squaring both sides and expanding we get $(a_x + b_x)^2 + (a_y + b_y)^2 = (c_x + d_x)^2 + (c_y + d_y)^2$ which implies $a_x^2 + 2a_xb_x + b_x^2 + a_y^2 + 2a_yb_y + b_y^2 = c_x^2 + 2c_xd_x + d_x^2 + c_y^2 + 2c_yd_y + d_y^2$. Regrouping we see that $|A|^2 + |B|^2 + 2(a_xb_x + a_yb_y) = |C|^2 + |D|^2 + 2(c_xd_x + c_yd_y)$. Since A, B, C and D are all of length 1, then $2(a_xb_x + a_yb_y) = 2(c_xd_x + c_yd_y)$. Note that $A \cdot B = (a_xb_x + a_yb_y) = |A||B|\cos(\theta_{AB})$ and $C \cdot D = (c_xd_x + c_yd_y) = |C||D|\cos(\theta_{CD})$. Thus $\theta_{AB} = \pm\theta_{CD}$. Similarly, we see that $\theta_{AD} = \pm\theta_{BC}$. Since $360 = \theta_{AB} + \theta_{BC} + \theta_{CD} + \theta_{DA} = 2\theta_{AB} + 2\theta_{BC}$. We see that θ_{AB} and θ_{BC} are supplementary. Similarly θ_{BC} and θ_{CD} are supplementary. Hence the Diagonals of $ABCD$ are each of length 2 implying $ABCD$ is a rectangle.

If $ABCD$ is a rectangle inscribed in the unit circle, then the the line connecting the midpoint of the segment connecting A and B to the midpoint of the segment connecting C and D goes through the center of the rectangle which is $(0, 0)$. The midpoints are $(A + B)/2$ and $(C + D)/2$ respectively. Since $|(A + B)/2| = |(C + D)/2|$, then $(A + B)/2 = -(C + D)/2$ or $A + B + C + D = (0, 0)$.