UNM - PNM STATEWIDE MATHEMATICS CONTEST L

November 3-6, 2017 First Round Three Hours

1. Let A, B be two sets of integers. $A = \{2, 0, 1, 7\}$ and $B = \{x \mid -x \in A,\ 2 - x^2 \notin A\}$. Find the sum of all the integers in B.

   Answer: -9.

   Solution: It is easy to see that $B \subset \{-2, 0, -1, -7\}$. When $x = 0, -1;\ 2 - x^2 \in A$ and when $x = -2, -7;\ 2 - x^2 \notin A$. Thus $B = \{-2, -7\}$. Hence the sum of B is -9.

2. Right now my age is the sum of the ages of my two sisters . Two years ago I was twice as old as my middle sister. Eight years from now my younger sister will be the age I am now. How old am I now?

   Answer: 14

   Solution: Let $m =$ middle sister's current age, $y =$ younger sister's current age, $x =$ my current age. We have
   (1) $m + y = x$
   (2) $y + 8 = x$, or $y = x - 8$
   (3) $2(m - 2) = x - 2$, or $m = \frac{1}{2}x + 1$
   Substituting yields $\frac{1}{2}x + 1 + x - 8 = \frac{3}{2}x - 7 = x$, or $\frac{1}{2}x = 7$. The ages are $6, 8$, and 14.

   We can also use matrices to solve this problem. As above my age is x, my middle sister's age is m and my younger sister's age is y. Then the three equations above correspond to the following system of equations:

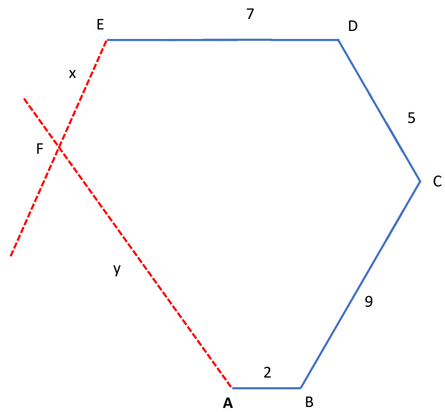
$$\begin{aligned} x - m - y &= 0 \\ x - 2m &= -2 \\ x - y &= 8 \end{aligned}$$

   We write the system in augmented matrix form $\begin{pmatrix} 1 & -1 & -1 & | & 0 \\ 1 & -2 & 0 & | & -2 \\ 1 & 0 & -1 & | & 8 \end{pmatrix}$ then we row reduce to obtain first the matrix $\begin{pmatrix} 1 & -1 & -1 & | & 0 \\ 0 & -1 & 1 & | & -2 \\ 0 & 1 & 0 & | & 8 \end{pmatrix}$ then the matrix $\begin{pmatrix} 1 & -1 & -1 & | & 0 \\ 0 & 1 & 0 & | & 8 \\ 0 & 0 & 1 & | & 6 \end{pmatrix}$
   Using back substitution, we see that my current age must be 14.

3. Four consecutive sides of an equiangular hexagon have lengths 2, 9, 5, and 7. Find the lengths of the two remaining sides.

   Answer: 4 and 10.

   Solution: Let $ABCDEF$ be the hexagon with side lengths 2, 9, 5, 7, x and y.



Extending the sides $\overline{AF}$, $\overline{BC}$ and $\overline{DE}$ we form three small equilateral triangles ABX, CDY and EFZ and one large equilateral triangle XYZ. The length of XY is $2+9+5 = 16$ which is also the length of YZ and XZ. $16 = YZ = 5 + 7 + x$, which implies $x = 4$ and $16 = XZ = 4 + y + 2$ which implies $y = 10$.

4. Find all the prime numbers p such that $4p^2 + 1$ and $6p^2 + 1$ are prime numbers.
   Answer: 5.
   Solution: When $p \equiv \pm 1 \pmod 5$, $5|4p^2 + 1$. When $p \equiv \pm 2 \pmod 5$, $5|6p^2 + 1$. Since $4 \cdot 5^2 + 1 = 101$ and $6 \cdot 5^2 + 1 = 151$ and these are both primes, then p must be 5.
5. Kayla loves almond butter but cannot afford to buy her own. Her roommate Diane bought a 200 mL jar. Kayla decides to steal some almond butter from Diane's jar every day immediately after Diane eats, but to make sure that Diane doesn't know, she will steal at most 20 mL but no more than one quarter of what is left in the jar. If Diane eats 30 mL each day or finishes the jar, how much almond butter does Kayla steal?

   Answer: 63.125=505/8 mL
   The jar starts with 200 mL and Diane eats 30 mL, leaving 170 mL left. Since 1/4 is greater than 20 mL, Kayla eats 20 mL leaving 150 mL. The second day Diane again eats 30 mL, leaving 120 mL. Since 1/4 is greater than 20 mL, Kayla eats 20 mL leaving 100 mL. The third day Diane again eats 30 mL, leaving 70 mL. However, now 1/4 of the jar is 35/2=17.5 mL and Kayla eats 35/2=17.5 mL leaving 105/2=52.5 mL. The fourth day Diane eats 30 mL, leaving 45/2=22.5 mL and 1/4 of the jar is 45/8=5.625 mL which Kayla will then eat leaving 16.875 mL which Diane will eat on the 5th day. In total, Kayla eats $20 + 20 + 17.5 + 5.625 = 63.125$ mL.

6. Consider the table:

1	2	3						
4	5	6	7	8	9			
10	11	12	13	14	15	16	17	18

If we continue to add rows so that the next row has 3 more consecutive terms than our current row, in which row is 2017?

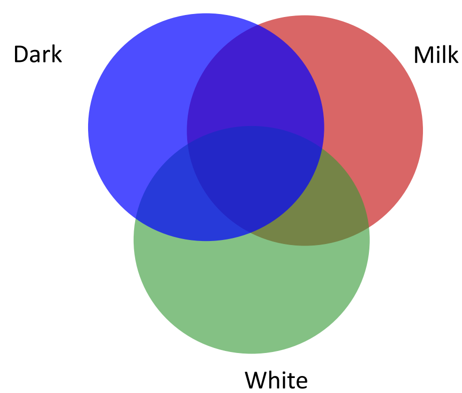
   Answer: The 37th row.

Solution: The nth row will contain $\dfrac{3n(n-1)}{2}+1, \dfrac{3n(n-1)}{2}+2, \ldots \dfrac{3n(n-1)}{2}+3n = \dfrac{3n(n+1)}{2}$. So we are on the search for n such that $\dfrac{3n(n-1)}{2} < 2017 \leq \dfrac{3n(n+1)}{2}$. Since $\dfrac{3 \cdot 37 \cdot 36}{2} = 1998$ and $\dfrac{3 \cdot 38 \cdot 37}{2} = 2109$, we see that 2017 is in the 37th row.

7. Two hundred people have come to hear a lecture on the benefits of chocolate. A survey is given out at the beginning as to the chocolate preferences of the audience members. Suppose 120 people like dark chocolate, 100 people like milk chocolate, 60 people like white chocolate, 40 people do not like any kind of chocolate and 10 people like all three kinds of chocolate, how many people like at least 2 types of chocolate?

Answer: 110

Solution: There are $200-40=160$ people who like chocolate of some sort. Inclusion-exclusion says that $160 = 120+100+60-(|Dark \cap Milk| + |Dark \cap White| + |Milk \cap White|) + 10$. Solve, we have $|Dark \cap Milk| + |Dark \cap White| + |Milk \cap White| = 130$.

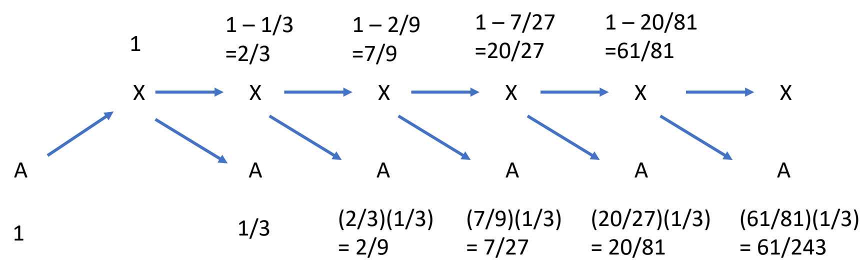


If we look at the Venn diagram (see figure), we see that what we want is the area where the circles intersect. The sum of the pairwise intersections overcounts the central part by two, so we need to subtract 20 from our partial answer. This gives $|Dark \cap Milk| + |Dark \cap White| + |Milk \cap White| - 20 = 110$.

8. Every week Iris chooses a fruit of the week from the following fruits: apple, banana, orange and mango. She decides that each week, she will pick a fruit which is different than previous week's fruit and she will choose one randomly from the remaining three. Now suppose in the first week of the school year, Iris chose apple. What's the probability that she will pick apple again in the seventh week of the school year? Provide your answer in the simplest form.

Answer: $\frac{61}{243}$.

Solution: A tree diagram solution:



A recursive solution: Let P_k be the probability that Iris will pick apple as her fruit of the week in the k^{th} week. Then we have

$$P_{k+1} = \frac{1}{3}(1 - P_k).$$

Thus we have

$$P_{k+1} - \frac{1}{4} = -\frac{1}{3}(P_k - \frac{1}{4}).$$

So

$$P_7 - \frac{1}{4} = \left(-\frac{1}{3}\right)^6 (P_1 - \frac{1}{4}) = \frac{1}{243 * 4}.$$

$$P_7 = \frac{61}{243}.$$

Another solution: We can model the system as the set A (apple) and X (everything else). If we start in state A, we are guaranteed to end up in X the next week, and if we start in X, we will end up in A one-third of the time and (in the other two parts of) X two-thirds of the time the next week. In linear algebra terms, we can say that, if A currently has probability p, and X has probability $q = 1 - p$, represented by vector $\begin{pmatrix} p \\ q \end{pmatrix}$, then the next week's state has probability $M \cdot \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & \frac{2}{3} \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix}$. Note that we can express the matrix M as $\frac{1}{3} \cdot \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$

We want to compute $M^6 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

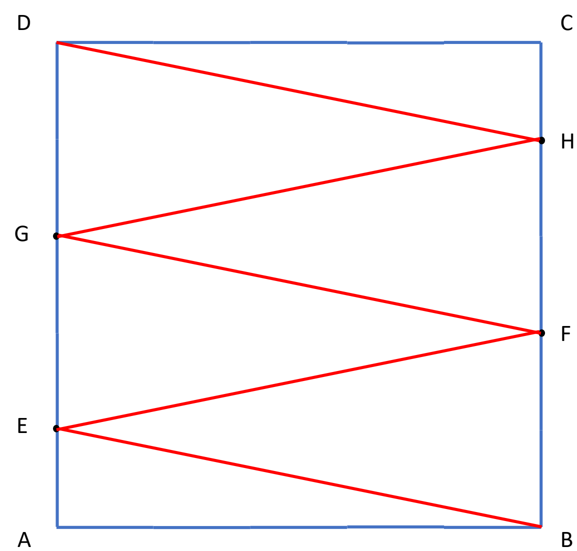
We have $M^3 = \frac{1}{27}\begin{pmatrix} 6 & 7 \\ 21 & 20 \end{pmatrix}$, and squaring this yields $M^6 = \frac{1}{729}\begin{pmatrix} 183 & 182 \\ 546 & 547 \end{pmatrix}$. Multiplying this against the initial vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ yields
$\frac{1}{729}\begin{pmatrix} 183 \\ 546 \end{pmatrix} = \frac{1}{243}\begin{pmatrix} 61 \\ 182 \end{pmatrix}$. Note that the sum of the two final probabilities is 1 (a sanity check!). The probability of ending in state A (apple) is $\frac{61}{243}$.

9. On square $ABCD$, points E and G are placed on $\overline{AD}$ so that G is between E and D and F and H are placed on $\overline{BC}$ so that F is between B and H and the lengths $BE = EF = FG = GH = HD = 52$ units. Find the area of $ABCD$.

Answer: 2600 units squared.

Solution: Ans: Area$(ABCD) = 2600$.

(see figure, below) From the figure, we see that one of the lengths is equal to $\sqrt{1^2 + 5^2} \cdot \frac{s}{5}$, where s is the side of the square. So, $s = \frac{5}{\sqrt{26}} \cdot 52$, and the area of the square is then $\frac{25}{26} \cdot 52^2 = 25 \cdot 2 \cdot 52 = 2600$.



Note that triangles BEF, EFG, FGH and GHD are congruent isosceles triangles so the angles $\alpha = \angle EBF = \angle BFE = \angle GEF = \angle EGF = \angle HFG = \angle FHG = \angle DGH = \angle GDH$. Since $\overline{AD}$ is parallel to $\overline{BC}$ and $\angle GDH$ and $\angle CHD$ are alternate interior angles as are $\angle FBE$ and $\angle AEB$ we can also conclude that $\angle AEB = \angle CHD = \alpha$. If we insert all the altitudes of these isosceles triangles, we see that we can divide the square into 10 congruent right triangles each having bases of length a (the length of a side of the square $ABCD$), height of length $\dfrac{a}{5}$ and hypotenuse of length 52 units. Using the Pythagorean

Theorem, we see that $a^2 + \frac{a^2}{5^2} = (52)^2$ or $\frac{25(52)^2}{26} = a^2$ which simplifies to $a^2 = 2600$ units squared which is the area we are searching for.

10. Let a, b be real numbers and $f(x) = ax+b$. Suppose for any $x \in [0,1]$, we have $|f(x)| \leq 1$. Find the maximum value of ab.

 Answer: $\frac{1}{4}$.

 Solution: Since $f(1) = a+b$ and $f(0) = b$, we have $ab = (f(1) - f(0))f(0)$. Notice that

$$\begin{aligned}(f(1) - f(0))f(0) &= -\left(f(0) - \frac{1}{2}f(1)\right)^2 + \frac{1}{4}(f(1))^2 \\ &\leq \frac{1}{4}(f(1))^2 \\ &\leq \frac{1}{4}.\end{aligned}$$

 The maximum value of ab can be archived for $a = b = \pm\frac{1}{2}$.

We thank Bill Cordwell for several nice pictures as well as some additional solutions to the problems.