UNM - PNM STATEWIDE MATHEMATICS CONTEST XLVIII

February 6, 2016 First Round Three Hours

1. Suppose there are 9 lights arranged on tic tac toe board so that one is in each square. Suppose further that there are six light switches one for each row and column. Flipping any of these switches turns on all lights that are off and turns off all lights that are in the column/row controlled by this switch. If there is exactly one light on, can you turn all the lights on using the given switches? As in all problems you need to explain your answer.

   Solution: Let us consider the problem on a 2 by 2 board. Starting with 1 light on and 3 off, a direct inspection shows that flipping a switch will result in either 1 light on and 3 lights off, or 3 lights on and 1 light off.

1	0
0	0

$\longrightarrow$

0	1
0	0

or

0	0
1	0

or

1	1
0	1

or

1	0
1	1

   Notice that the difference between the lights that are on and the lights that are off is 2. This is an invariant preserved after flipping any switch. Thus, no matter how we flip the switches we cannot have them all on.

   The 3 by 3 case can be reduced to the 2 by 2 case by selecting a 2 by 2 board, which includes the light that is on, from the 3 by 3 one (select four boxes that are in two fixed columns and in two fixed rows including the box with the light on). Flipping the switches in the remaining row or column does not change the lights in the picked 2 by 2 square. Since we cannot turn all lights on in the 2 by 2 square we cannot turn them all on in the 2 by 3 square.

0	0	0
0	0	1
0	0	0

2. A student is offered two different after school jobs. One pays \$10 an hour, the other pays \$100 for the first hour, but the hourly rate decreases by half for each additional hour worked. What are the maximum number of hours the student can work at the second job, so that his total earnings in the second job are more than the total earnings in the first? Note: You are asked to find the maximum full hours of work (i.e. integer number) that maximize the earnings.

   Solution: In the job paying \$10 an hour, if the student works for n hours he will earn $10n$ dollars. In the other job, he will earn

$$100 + 50 + \cdots 100/2^{n-1} = 100\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right) = \frac{100(1 - \frac{1}{2^n})}{1 - \frac{1}{2}} = \frac{200(2^n - 1)}{2^n}$$

   dollars. So we need to determine n so that

$$\frac{200(2^n - 1)}{2^n} > 10n$$

   i.e., $200(2^n - 1) > 10n2^n$ and $200 > 10n$ so $n < 20$. The maximum n is 19. In the first job the student will earn 190 dollars and in the second the student will earn $200 - 200/2^{19}$ dollars which is just a little bit less than 200 dollars.

3. A school purchased 4 peach trees, 4 apricot trees and 6 cherry trees that they want to plant in a row on the school grounds. If the trees are planted in random order, what is the probability that no two cherry trees are planted next to each other?

Solution: The peach and apricot trees can be thought of as a single type of trees, i.e., we can treat them as indistinguishable since either of them is a "good" divider of the cherry trees. Thus we will think of the peach and apricot trees as dividers for the cherry trees. Since we have 8 such dividers we have 9 places where we could place the cherry trees. As there are 6 cherry trees there are $C(9,6) = \frac{9\cdot 8\cdot 7}{3!} = 84$ ways of placing the cherry trees so that none touch. Here $C(9,6)$ denotes the number of ways we can choose 6 objects out of 9 distinct objects. On the other hand, since there are a total of 14 trees there are $C(14,6) = \frac{14\cdot 13\cdot 12\cdot 11\cdot 10\cdot 9}{6!} = 3003$ ways of placing the cherry trees without any conditions. Thus, the probability that no two cherry trees are planted next to each other is $\frac{84}{3003} = \frac{12}{429} = \frac{4}{143}$.

4. A farmer's house is in the shape of a convex pentagon with perimeter P and area A. The yard around the house includes all points that are at a distance at most $20m$ from the house. Find the area of the farmer's lot (yard plus house).

   Solution: Let us describe all points in the yard. Clearly along every side of the house we should add a rectangular area of width $20m$. On the other hand, at each of the corners of the house the yard will be a sector of radius 20 between the sides of length 20 along the adjacent sides. Alternatively, we can describe these sectors as being sectors of radius 20 and opening complementary to 180^o of the interior angle of the house at the corresponding corner (vertex). As the sum of the interior angles α_i of a convex pentagon is $\alpha_1 + \cdots + \alpha_5 = 3\cdot 180^o$ the yard sectors will have total opening of $(180^o - \alpha_1) + \cdots + (180^o - \alpha_5) = 5\cdot 180^o - 3\cdot 180^o = 360^o$, i.e., a whole disc of radius 20. Therefore, the lot will be $400\pi + 20\cdot P + A\, m^2$corresponding to the yard at the corners, the 5 rectangles along each of the sides and the interior of the house.

5. Show that if 19 points are chosen on a square of side of length 1 then there is a triangle with vertices among these points whose area is at most $\frac{1}{18}$.

   Solution: Divide the square in nine equal squares using lines parallel to the sides. At least one of these squares will have three of the chosen points.

   We have reduced the problem to showing that from a square of sides of length $1/3$ we can cut a triangle of area at most $1/18$. To see this pick one of the points and draw a line through it parallel to two of the opposite sides of the square. In this way we split the square in two rectangles of sides x by $1/3$ and y by $1/3$ with $x+y = 1/3$. Our triangle can have at most half of the area of each of the obtained rectangles, so the total area is less than $\frac{1}{2}\cdot\frac{1}{3}\cdot x + \frac{1}{2}\cdot\frac{1}{3}\cdot = \frac{1}{6}(x+y) = \frac{1}{18}$.

6. For a positive integer k let $\sigma(k)$ be the sum of the digits of k. For example, $\sigma(1234) = 1+2+3+4 = 10$ while $\sigma(4) = 4$. Let $a_1 = 2016^{2016}$ and define $a_{n+1} = \sigma(a_n)$, $n = 1, 2, 3, \ldots$. Find a_5.

   Solution: One of the keys here is that $\sigma(k)$ and k have the same remainder when divided by 9. Now, 2016 is divisible by 9 since $\sigma(2016) = 9$ is divisible by 9. Therefore, every power of 2016 is also divisible by 9. Therefore, if we can see that a_5 is a single digit number then $a_5 = 9$ as this is the only digit divisible by 9. Next we turn to this task.

   Notice that $a_1 < 10,000^{2016} = 10^{4\cdot 2016} = 10^{8064}$, hence a_2 has less than 8064 digits which shows that $a_2 = \sigma(a_1) \le 9\cdot 8064 < 99\cdot 1000 < 99,000$. Thus $a_3 < 9\cdot 5 = 45$. This implies $a_4 = \sigma(a_3) \le 12$ (39 has the largest sum of its digits among the integers between 1 and 45). Thus a_5 has a single digit, which must be 9 as we already observed.

7. For a positive integer n let $S(n)$ denote the function which assigns the sum of all divisors of n. Show that if m and n are relatively prime positive integers then $S(mn) = S(m)\,S(n)$. For example, $S(6) = 1+2+3+6 = 12$, $S(2) = 1+2 = 3$ and $S(3) = 1+3 = 4$, so $S(6) = S(2)\,S(3)$, noting that 2 and 3 are relatively prime integers (they have no common divisor).

   Solution: By definition $S(n) = \Sigma_{d|n} d$-the sum of all divisors of n, where for integers a and b we write $a|b$ if a divides b. Let $a_1 = 1, a_2, \ldots, a_k = m$ be the different divisors of m and

$b_1 = 1, b_2, \dots, b_l = n$ all the divisors of b. Thus $\Sigma a_i = S(m) = a_1 + a_2 + \cdots + a_k$ and $\Sigma b_j = S(n) = b_1 + b_2, \cdots + b_l$. Now, consider $\Sigma a_i\, b_j = S(m)\, S(n) = (a_1 + a_2 + \cdots + a_k)(b_1 + b_2, \cdots + b_l)$. Clearly every element in this sum is a divisor of mn. Furthermore, since m and n are relatively prime it follows that if $d|mn$ then either $d = 1$ or d is a product of two integers one of which divides m, the other divides n and at least one of them is greater than 1. In other words d will appear in the sum $S(m)\, S(n) = \Sigma a_i\, b_j$ exactly once. This shows that $S(mn) = S(m)\, S(n)$.

8. Find all non-negative integer solutions of the equation $n(n + 1) = 9(m - 1)(m + 1)$.

   Answer: m=1, n=0 or m=3, n=8.

   Solution: Considering the given equation as a quadric w.r.t n, we need to solve $n^2 + n - 9(m^2 - 1) = 0$. The roots are given by

$$n = \frac{-1 \pm \sqrt{1 + 36(m^2 - 1)}}{2}.$$

   In particular, $1 + 36(m^2 - 1)$ is an exact square, so $1 + 36(m^2 - 1) = k^2$. Thus, we try to solve the equation

$$(6m)^2 = k^2 + 35.$$

   Since $35 = (6m)^2 - k^2 \geq (6m)^2 - (6m - 1)^2 = 12m - 1$ it follows that $1 \leq m \leq 3$ taking into account that $m = 0$ gives $n < 0$. Now, we can finish by considering the three possible cases of the original equation. For $m = 1$ we have $n = 0$, $m = 2$ gives $n(n + 1) = 27$ which has no solution, and $m = 3$ leads to $n(n + 1) = 9 \cdot 8$ hence $(n + 9)(n - 8) = 0$ hence $n = 8$.

   Another solution is obtained by completing the square (w.r.t. n) in the given equation is equivalent to $36m^2 - (2n + 1)^2 = 35$, i.e.,

$$(6m + (2n + 1))(6m - (2n + 1)) = 35.$$

   Taking into account that m and n are non-negative integers and $35 = 35 \cdot 1 = 7 \cdot 5$ we obtain the answer.

9. Suppose every point in the plane is colored by one of two given colors, say red or blue. Given a triangle Δ, show that there is a triangle in the colored plane whose vertices are of the same color and is similar to the given triangle Δ.

   Solution: First we show that there are three points of the same color such that one of the points is midway between the other two. Indeed, consider any two points that are of the same color, say blue. For ease of reference, think of taking the line through two blue points, call them 1 and 2; note that there are is at most one point of the same color then we are done using the remaining points. Now that we have the labeled points 1 and 2, look at the two points, call them 0 and 3, such that 1 is the midpoint between 0 and 2, and 2 is the midpoint between 1 and 3. If 3 is blue we are done. If 3 is red, consider the point 0. If 0 is blue, then 0, 1 and 3 are blue and we are done again. Finally, if 0 and 3 are red while 1 and 2 are blue, we consider the midpoint between 0 and 3, which is also the midpoint between 1 and 2. This point, which is either blue or red, together with either 0 and 3, or 1 and 2 will meet the wanted condition.

   Now that we know that there are three blue (otherwise we can rename the points and reduce to this case) points A, C' and B such that C' is the midpoint between the other two, take a point C such that the triangle $\triangle ABC$ which is similar to the given triangle Δ. Let B' and A' be the midpoints of the sides AC and BC respectively. Notice that each of the 5 triangles $\triangle ABC$, $\triangle A'B'C'$, $\triangle A'B'C$, $\triangle AB'C'$ and $\triangle A'BC'$ is also similar to the given triangle. Now, if any of the vertices of $\triangle B'A'C$ is blue, then one of the five triangles will be blue. In the other case, all vertices of $\triangle B'A'C$ are red, so we are done again.

10. Let P be a point on the triangle $\triangle ABC$ (inside or on the boundary). Let r_a, r_b and r_c be the distance from P to the sides BC, CA and AB, respectively.

a) Show that

$$r_a \cdot a + r_b \cdot b \leq |PC| \cdot c \quad \text{and also} \quad r_a \cdot b + r_b \cdot a \leq |PC| \cdot c,$$

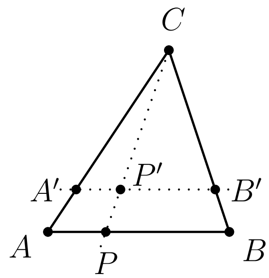
where $a = |BC|$, $b = |CA|$ and $c = |AB|$.

b) Assuming the inequalities of part a), show that

$$\frac{|PA| + |PB| + |PC|}{r_a + r_b + r_c} \geq 2.$$

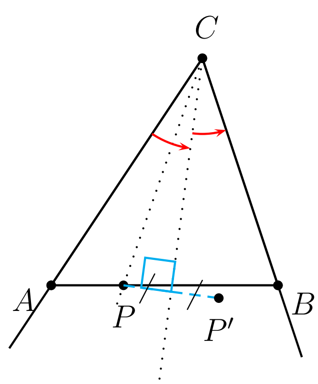
Solution: This is the so called Erdös-Mordell problem. Below is the solution given by V. Komornik, The American Mathematical Monthly, Vol. 104, No. 1 (Jan., 1997), pp. 57–60.

a)



Observe that it is enough to consider the case when P is on the side BC. Indeed, for P' inside the triangle, consider the point P which is the intersection of the line through C and P' and the side BC together with the points A' and B' on AC and BC, respectively, such that $A'B'$ is parallel to AB. From the similarity of the $\triangle A'B'C$ and $\triangle ABC$ the inequality for P' follows from the inequality for P (by dividing the latter by the coefficient of similarity) and conversely, the inequalities at P' imply those for P. In fact, this shows that if either of the inequalities holds for one point in the angle $\angle ACB$ (including the rays $\overrightarrow{CA}$ and $\overrightarrow{CB}$) then this inequality holds for all points in the angle $\angle ACB$.

Next we will consider the case when P is on the side BC. The inequality $r_a \cdot a + r_b \cdot b \leq |PC| \cdot c$ is actually obvious noting that the sum of the areas of $\triangle CPA$ and $\triangle CPB$ equals the area of $\triangle ABC$ while $|PC| \cdot c$ is at least twice the area of of $\triangle ABC$ with equality iff CP is the altitude through C. Thus the first inequality holds for all points in the angle $\angle ACB$.



Now, the second inequality, $r_a \cdot b + r_b \cdot a \leq |PC| \cdot c$, follows by considering the point P'- the reflection of P through the bisector of the angle $\angle ACB$. Notice that $r'_a = r_b$, $r'_b = r_a$ while

$|P'C| = |PC|$. Now, writing the first inequality for P' shows that the second inequality holds at P.

b) Recall the inequality $\frac{x}{y} + \frac{y}{x} = \frac{x^2+y^2}{xy} \geq 2$ for positive numbers x and y, which follows from $x^2 + y^2 - 2xy = (x-y)^2 \geq 0$ with equality iff $x = y$.

Now, from the second inequality of part a) and the above inequality we have

$$\begin{aligned} |PA| + |PB| + |PC| &\geq \frac{r_b \cdot c + r_c \cdot b}{a} + \frac{r_c \cdot a + r_a \cdot c}{b} + \frac{r_a \cdot b + r_b \cdot a}{c} \\ &= r_a \left(\frac{b}{c} + \frac{c}{a} \right) + r_b \left(\frac{c}{a} + \frac{a}{c} \right) + r_c \left(\frac{a}{b} + \frac{b}{a} \right) \geq 2 \left(r_a + r_b + r_c \right). \end{aligned}$$

Furthermore, it follows that equality is achieved iff $a = b = c$ and then P is the center of the equilateral triangle.