UNM - PNM STATEWIDE MATHEMATICS CONTEST XLVII

November 7-10, 2014 First Round Three Hours

1. In the solution of the following 6 by 6 sudoku puzzle, what are the numbers a, b, c and d. By a solution of the puzzle is meant a placement of the numbers 1 to 6 in the empty squares so that each row, each column and each 2x3 box contains the same number only once.

		2	5		
	3			1	
2					4
1	a	b	c	d	2
	2			3	
		5	4		

Solution: The only solution is (why?) as follows,

6	1	**2**	**5**	4	3
5	**3**	4	2	**1**	6
2	5	3	1	6	**4**
1	$a = 4$	$b = 6$	$c = 3$	$d = 5$	**2**
4	**2**	1	6	**3**	5
3	6	**5**	**4**	2	1

2. Simplify the expression

$$Q = \left(\frac{1}{(a+b)^2} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) + \frac{2}{(a+b)^3} \left(\frac{1}{a} + \frac{1}{b} \right) \right) a^2 b^2$$

Solution: $\frac{1}{(a+b)^2} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) + \frac{2}{(a+b)^3} \left(\frac{1}{a} + \frac{1}{b} \right) = \frac{a^2+b^2}{a^2b^2(a+b)^2} + \frac{2}{(a+b)^2 ab} = \frac{1}{a^2b^2}$, thus $Q = 1$.

3. Find the smallest positive integer with precisely 42 factors.

Solution: Any positive integer m can be expressed in the form $m = p_1^{n_1} \cdots p_r^{n_r}$ for distinct primes p_i. The number of factors is precisely $(n_1+1)(n_2+1)\cdots(n_r+1)$. If m has 42 factors then $(n_1+1)\cdots(n_r+1) = 42$. The factorizations of 42 are $2 \cdot 21$, $3 \cdot 14$, $6 \cdot 7$ and $2 \cdot 3 \cdot 7$. Consider $2^{20} \cdot 3 = 3145728$, $2^{13} \cdot 3^2 = 73728$, $2^6 \cdot 3^5 = 15552$ or $2^6 \cdot 3^2 \cdot 5 = 2880$. The smallest of which is 2880.

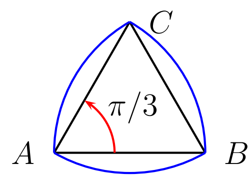
4. Find the exact value of $a = \sqrt{|12\sqrt{10} - 49|} - \sqrt{|12\sqrt{10} + 49|}$

Solution: Let $a = \sqrt{|12\sqrt{10} - 49|} - \sqrt{|12\sqrt{10} + 49|}$. First, notice that $a < 0$. Furthermore, $12\sqrt{10} - 49 < 0$ since $49^2 = 2401 > 1440 = \left(12\sqrt{10}\right)^2$. Thus, $a = \sqrt[2]{49 - 12\sqrt{10}} - \sqrt[2]{49 + 12\sqrt{10}}$.

$$\begin{aligned} a^2 &= 49 - 12\sqrt{10} + 49 + 12\sqrt{10} - 2\sqrt[2]{49^2 - \left(12\sqrt{10}\right)^2} = 2 \cdot 49 - 2\sqrt[2]{2401 - 1440} \\ &= 2 \cdot 49 - 2\sqrt[2]{961} = 2 \cdot (49 - 31) = 2 \cdot 18 = 36. \end{aligned}$$

Therefore $a = -6$.

5. Find the area of the figure which is the intersection of three discs of radius 1 with centers at the three vertices of an equilateral triangle with sides of length 1.



Solution: The area of a circular sector (wedge) with an opening angle $\alpha = \pi/3$ and radius $a = 1$ is $A_w = 1/2\alpha a^2 = \pi/6$ (1/6-th of the area of the unit disc). The area of the equilateral triangle with side $a = 1$ is $A_\triangle = 1/2a^2 \sin\alpha = 1/4\sqrt(3)$. The area A_s of a circular segment with opening angle $\alpha = \pi/3$ and radius 2 is therefore $A_s = A_w - A_\triangle = \pi/6 - 1/4\sqrt(3)$. Thus, the total area is then

$$A = 3A_s + A_\triangle = 1/2(\pi - \sqrt(3)).$$

6. The decimal $.37\overline{027}$ can be expressed in the form $\dfrac{m}{n}$ where m and n are relatively prime. Find $n - m$.

Solution:

$$.37\overline{027} = \frac{37}{100} + \frac{27}{99900} = \frac{37}{100} + \frac{1}{3700} = \frac{37^2+1}{3700} = \frac{1370}{3700} = \frac{137}{370}.$$

Thus $n - m = 370 - 137 = 233$.

7. Find the exact value $\sin\alpha$ if $\tan(\alpha/2) = \sqrt{3}/2$.

Solution: We will use that

$$\sin(\alpha + \beta) = \cos\alpha \sin\beta + \cos\beta \sin\alpha.$$

In particular, $\sin\alpha = 2\sin\alpha/2\cos\alpha/2$, hence $\sin\alpha = 2\tan\alpha/2\cos^2\alpha/2$. Now we write

$$1 + \tan^2(\alpha/2) = 1 + \frac{\sin^2\alpha/2}{\cos^2\alpha/2} = \frac{\cos^2\alpha/2 + \sin^2\alpha/2}{\cos^2\alpha/2} = \frac{1}{\cos^2\alpha/2},$$

hence $\cos^2\alpha/2 = \frac{1}{1+3/4} = 4/7$. This shows

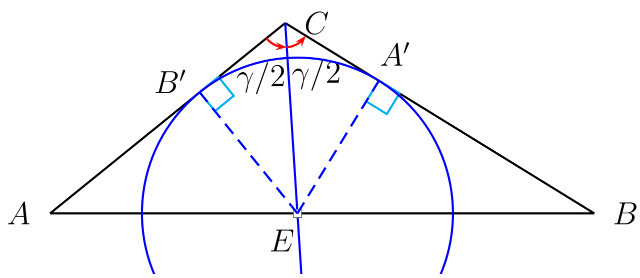
$$\sin\alpha = 2\tan\alpha/2\cos^2\alpha/2 = 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{4}{7} = \frac{4}{7}\sqrt{3}.$$

Remark: Recall Euler's formula $e^{i\alpha} = \cos\alpha + i\sin\alpha$ valid for any α. Euler's formula implies easily the identity $\sin(\alpha+\beta) = \cos\alpha\sin\beta + \cos\beta\sin\alpha$ by a comparison of the imaginary parts of $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$,

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

8. In a triangle with sides of lengths 5, 6 and 9 is drawn a circle touching the two shorter sides with a center on the longest side. Find the radius of the circle.

Solution:



I) One way to solve the problem is to use Heron's formula and express the area of the triangle $\triangle ABC$ in two ways. By Heron's formula the area of a triangle with sides of lengths a, b and c is

$$A = \frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

In our case we have $A = 10\sqrt{2}$. On the other hand, if R is the radius of drawn circle we have by considering the areas of $\triangle AEC$ and $\triangle BEC$ the formula $A = \frac{1}{2}R(6+5) = 11R/2$. Therefore, $11R/2 = 5\sqrt{2}$, hence

$$R = \frac{20\sqrt{2}}{11}.$$

II) We can solve the problem using trig functions and expressing the area in two ways, first as above and the second using the formula $A = \frac{1}{2}ab\sin\gamma$ for the area of a triangle with sides of lengths a and b with angle γ between them. Thus, we have $A = \frac{30\sin\gamma}{2} = 11R/2$, i.e.,

$$R = \frac{30\sin\gamma}{11}.$$

Now, by law of cosines, $c^2 = a^2 + b^2 - 2ab\cos\gamma$, we have $81 = 36 + 25 - 2\cdot 30\cdot\cos\gamma$, i.e., $\cos\gamma = 1/3$. Therefore, $\sin\gamma = \sqrt{1-1/9} = 2\sqrt{2}/3$. Using the area identity above it follows

$$R = \frac{20\sqrt{2}}{11}.$$

9. An urn contains black and white balls. It is known that we can draw at random with equal probability two balls of same color or two balls of different color. What is the number of balls that must be in the urn so that there are at least 2014 and at most 2114 balls.

Solution: Suppose there are n balls of which b black balls and $w = n - b$ white balls.

pair	prob. for first color	prob. for second color	overall prob.
(B,W)	$\frac{b}{n}$	$\frac{w}{n-1}=\frac{n-b}{n-1}$	$\frac{b(n-b)}{n(n-1)}$
(W,B)	$\frac{w}{n}=\frac{n-b}{n}$	$\frac{b}{n-1}$	$\frac{b(n-b)}{n(n-1)}$
(B,B) or (W,W)			$1-2\frac{b(n-b)}{n(n-1)}$

The probability of drawing two balls of different color is $2\frac{b(n-b)}{n(n-1)}$ so the (remaining) probability of drawing two balls of the same color is $1-2\frac{b(n-b)}{n(n-1)}$, see table above. Therefore we need to solve the equation

$$2\frac{b(n-b)}{n\,(n-1)} \;=\; 1-2\frac{b(n-b)}{n\,(n-1)}.$$

The above equation can be solved as follows after simplifying to equivalent equations (notice that $n\neq 0,\,1$),

$$\begin{aligned} 4bn-4b^2 &= n^2-n \\ n^2-4bn+4b^2 &= n \\ (n-2b)^2 &= n, \end{aligned}$$

hence the solution is $b=\frac{1}{2}n+\frac{1}{2}\sqrt{n}$ or $b=\frac{1}{2}n-\frac{1}{2}\sqrt{n}$. It follows that n is an exact square since b is an integer. Therefore the solution is $n=45^2=2025$ taking into account

$$44^2=1936<2014<45^2=2025<2125<2114<46^2=2116.$$

10. Consider the two arithmetic sequences
    (a) 17, 21, 25, 29, ... and (b) 16, 21, 26, ...
    Find the sum of the smallest 40 numbers that appear in both sequences.

Solution: $a_n=17+4\,(n-1)\,, b_n=16+5\,(n-1)\,$. In order to find the common numbers we need to solve $a_n=b_k$,i.e., $17+4\,(n-1)=16+5\,(k-1)$. The latter simplifies to

$$5k-4n=2.$$

Thus, our goal is to find all pairs of positive integers (k,n) such that $5k-4n=2$. For $n=1$ (resp. $k=1$) there is no positive integer k (resp. n) with $5k-4n=2$. However, $k'=n'=2$ is a solution which shows that the smallest common number of the two sequences is 21. Notice that if we take two solutions (k,n) and (k',n') of $5k-4n=2$ then their difference $(u,v)\stackrel{def}{=}(k-k',n-n')$ satisfies the homogeneous equation $5u-4v=0$. The latter can be solved by letting $v=5s$, hence $u=4s$, where s can be an arbitrary integer. By what we have already observed it follows that the pairs of positive integers which solve $5k-4n=2$ are $k=2+4s$, $n=2+5s$, $s=0,1,2,...$Thus, the common numbers, in increasing value, of the two sequences are

$$c_s=a_{2+5s}=21+20s, s=0,1,2,...$$

The sum of the first 40 common numbers is

$$S=\sum_{s=0}^{39}(21+20s)=21\cdot 40+20\cdot\sum_{s=0}^{39}s=21\cdot 40+20\cdot\frac{39\cdot 40}{2}=16440$$