UNM - PNM STATEWIDE MATHEMATICS CONTEST XLV

February 4, 2013 Second Round Three Hours

1. A fox running at its top speed can run 200m in 10sec, while a squirrel running at its top speed can run 300m in 50sec. The fox sees the squirrel 21 meters ahead and starts chasing it. Suppose that both the fox and the squirrel are running along a straight line in the same direction maintaining their respective top speeds throughout the chase. When will the fox catch the squirrel?

   Solution: From the given data it follows the fox is running at a speed of $200m/10sec = 20\,m/sec$, while the squirrel's speed is $6\,m/sec$. A quick way to find the answer is to note that the quantity we are looking for is relative - it depends only on the difference of the speeds of the fox and the squirrel. Thus, we can "slow down" everyone by the same speed. The problem is then equivalent to having the fox run at $14\,m/sec$ while the squirrel is stationary. Hence, it will take $21/14 = 3/2 = 1.5$ seconds for the fox to catch the squirrel.

   Alternatively, if t is the sought time we have that the fox will run $20m/sec \cdot t\,sec = 20t\,m$ in the direction of the chase, while the squirrel will run $6m/sec \cdot t\,sec = 6t\,m$. The difference between these two distances is $21\,m$, i.e., we have

$$20t - 6t = 21, \quad \text{hence} \quad t = 21/14 = 3/2.$$

2. Solve the equation $(6x-1)^2(x+1) - (6x+1)^2(x-1) = 14$.

   Solution: After expanding and then factoring we find

$$(6x-1)^2(x+1) - (6x+1)^2(x-1) - 14 = 48x^2 - 12 = 12(2x-1)(2x+1).$$

   Thus, there are two solutions of the given equation $x = \pm 1/2$.

3. For which positive integers n is the triple $\left(\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}\right)$ the sides of a triangle.

   Solution: First, we observe that $\frac{1}{n} > \frac{1}{n+1} > \frac{1}{n+2}$. Hence, by the triangle inequality, the condition

$$\frac{1}{n} < \frac{1}{n+1} + \frac{1}{n+2}$$

   holds iff the triple of given numbers is the lengths of the sides of a triangle. Notice that we use that the other two triangle inequalities follow taking into account the observed order between the lengths of the sides. Thus, we need to solve the equivalent inequlaity

$$\frac{2n+3}{(n+1)(n+2)} > \frac{1}{n}.$$

   Cross multiplying we see that this inequality holds if and only if $2n^2 + 3n > n^2 + 3n + 2$ or when $n^2 > 2$. Hence, the answer is $n > 1$.

4. In a volleyball tournament every team plays one game against each of the remaining teams. Show that regardless of the set schedule, at every moment of the tournament there are at least two teams who have played the same number of games.

   Solution: Let n be the number of teams in the tournament. Suppose the claim is not true, so there is a moment when all teams have played different number of games. Since each team can play at most $n-1$ games and all of them have played different number of games there is a bijection between the number of teams and the integers 0, 1, ..., n-1 which is defined by assigning to each team the number of plaid games. This is a contradiction since the team with maximum possible plaid games must have plaid n-1 games, while the team with least games must have plaid no game.

5. To every pair of positive real numbers x and y, we assign a positive real number $x * y$ satisfying the two properties that $x * x = 1$ and $x * (y * z) = (x * y) \cdot z$ where $\cdot$ represents the standard multiplication of real number. Determine $61 * 2013$.

   Solution: First note that if we substitute $z = y$, we obtain

$$x * 1 = x * (y * y) = (x * y) \cdot y.$$

   Hence,

$$x * y = (x * 1) \div y. \tag{1}$$

   Now substitute $y = z = x$. We obtain $x * 1 = x * (x * x) = (x * x) \cdot x = 1 \cdot x = x$. Replacing $x * 1$ with x in equation (1) we obtain that $x * y = x \div y$ for all positive real numbers x and y. Thus $61 * 2013 = \dfrac{1}{33}$.

6. (Barbeau, Klamkin, Moser, *Five hundred mathematical challenges*. Spectrum MAA, 1995) Show that there is no solution to THREE + FIVE = EIGHT if different letters denote different digits.

   Solution: Since we are adding a four digit number to a five digit number and the answer is a five digit number we must have T+1=E. Next we look at the units. We have either 2E=T or 2E=10+T depending on whether (the sum of the units) 2E is less or greater or equal than 10. In the first case, 2E=E-1, which is not possible. Hence, the second case holds, i.e., E=9, T=8. Expanding the given numbers in base 10, the given equality can be written as

$$10^4T + 10^3H + 10^2R + 10E + E + 10^3F + 10^2I + 10V + E = 10^4E + 10^3I + 10^2G + 10H + T,$$
$$10^4(T - E) + 10^3(H + F - I) + 10^2(R + I - G) + 10(E + V - H) + 2E - T = 0,$$
$$10^3(H + F - I) + 10^2(R + I - G) + 10(9 + V - H) + 10 = 10^4,$$

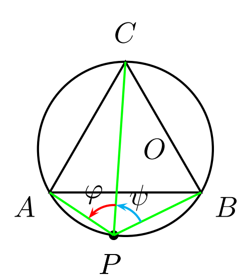
   taking into account the determined values of T and E. Thus we have

$$10^3(H + F - I) + 10^2(R + I - G) + 10(V - H) = 10^4 - 10^2,$$
$$10^2(H + F - I) + 10(R + I - G) + (V - H) = 10^3 - 10.$$

   This shows that $10|(V - H)$ and since both are between 0 and 9 it follows V=H. This is a contradiction since by assumption different letters represent different digits.

7. Show that the sum of the squares of distances from a point on the circumscribed circle of an equilateral triangle is independent of the point. NOTE: The same is true for the forth powers, but you don't need to show it.

   Solution: Let a, b and c be the distances from P to the vertices A, B and C, respectively. Let m be the length of the sides of the equilateral triangle. Suppose P is a point on the arc of the circumscribed circle opposite the vertex C.



   By the law of cosines we have

$$\begin{aligned} m^2 &= a^2 + c^2 - 2ac\cos\varphi = a^2 + c^2 - ac \\ m^2 &= b^2 + c^2 - 2bc\cos\psi = b^2 + c^2 - bc \\ m^2 &= a^2 + b^2 - 2ab\cos(\varphi + \psi) = a^2 + b^2 + ab, \end{aligned}$$

since $\varphi = \psi = 60^o$, using that the given triangle is equilateral, and $\cos 60^o = -\cos 120^o = 1/2$. First, we will show that $c = a + b$. Adding the first two of the above identities and multiplying the last one by 2 we obtain the equations

$$\begin{aligned} 2m^2 &= a^2 + b^2 + 2c^2 - c(a+b) \\ 2m^2 &= 2a^2 + 2b^2 + 2ab. \end{aligned} \tag{2}$$

Thus, we have

$$a^2 + b^2 + 2c^2 - c(a+b) = 2a^2 + 2b^2 + 2ab$$

which implies

$$\begin{aligned} 0 &= (a+b)^2 + c(a+b) - 2c^2 = (a+b)^2 - c^2 + c(a+b) - c^2 \\ &= (a+b-c)(a+b+c) + c(a+b-c) = (a+b-c)(a+b+2c). \end{aligned}$$

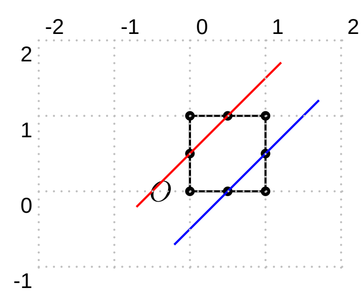
Hence $c = a + b$. Now, the first equation in (2) gives

$$2m^2 = a^2 + b^2 + 2c^2 - c(a+b) = a^2 + b^2 + c^2 + c(c-a-b) = a^2 + b^2 + c^2,$$

which is what we wanted to show.

8. Find all non-negative integer numbers n and m such that $3^n = 1 + 2^m$.
   Solution: When $m = 0$ there is no solution, while $m = 1$, $n = 1$ is a solution. Another solution is $m = 3$, $n = 2$. Next we prove that there are no other solutions. Working mod 4 we have $3^n - 1 = 2$ and $3^n - 1 = 0$ when n is odd or even respectively. On the other hand for $m \geq 2$ we have $4|2^m$. Thus we only have to consider the case when n is even. Let $n = 2k$, so we need to solve $2^m = 3^{2k} - 1 = (3^k - 1)(3^k + 1)$. Since $3^k - 1$ and $3^k + 1$ are two successive even numbers their gcd is equal to 2. For their product to be then a power of 2 both of the numbers $3^k - 1$ and $3^k + 1$ are powers of 2. Hence, we must have $3^k - 1 = 2$, hence $k = 1$ or $n = 2$.

9. Find all points $P(x, y)$ in the unit square $R = [0, 1] \times [0, 1]$ such that the difference between the two coordinates is at most 1/2, i.e., $|x - y| \leq 1/2$.
   Solution: Notice that if $x \geq y$ then $|x - y| = x - y$ while if $x \leq y$ then $|x - y| = y - x$. The points with desired property for which $x \geq y$ are the points between the line $y = x - 1/2$ (blue line below) and the parallel to it diagonal. Indeed the points "above" the line $y = x - 1/2$ satisfy $y > x - 1/2$, i..e, $x - y < 1/2$ while the points below the diagonal satisfy $x > y$. On the other hand the wanted points for which $x \leq y$ are the points between the line $y = x + 1/2$ (red line below) and the diagonal parallel to it. Thus all points with the desired property are the points in the unit square between the lines $y = x \pm 1/2$ (the red and blue lines).



10. Two numbers are picked at random from the interval $(0, 1)$. What is the probability that the difference between the two numbers is at most 1/2?
    Solution: The probability is the ratio of the area of the region in the last problem and the area of the unit square. This is $1 - 2 \times (1/2) \times (1/2)^2 = 3/4$.