UNM - PNM STATEWIDE MATHEMATICS CONTEST XLV

November 2-5, 2012 First Round Three Hours

1) Find the closest integer to $a+b$ if

$$a = \frac{1}{4} + \frac{3}{8} + \frac{5}{12} + \cdots \frac{1005}{2012}$$

and

$$b = \frac{5}{8} + \frac{7}{12} + \frac{9}{16} + \cdots \frac{1009}{2016}.$$

Answer: 503.

Solution: Adding the two numbers and grouping the terms as shown we compute

$$\begin{aligned} a+b &= \frac{1}{4} + \left(\frac{3}{8} + \frac{5}{8}\right) + \left(\frac{5}{12} + \frac{7}{12}\right) + \ldots + \left(\frac{1005}{2012} + \frac{1007}{2012}\right) + \frac{1009}{2016} \\ &= \frac{1}{4} + 502 + \frac{1009}{2016} = 502 + \frac{1}{4} + \frac{1009}{2016}. \end{aligned}$$

We used that $\left(\frac{3}{8} + \frac{5}{8}\right) = \left(\frac{5}{12} + \frac{7}{12}\right) = \ldots = \left(\frac{1005}{2012} + \frac{1007}{2012}\right) = 1$ and there are $(2012/4) - 1$ such terms. Now, since $\frac{1009}{2016} > \frac{1}{4}$ it follows that $\frac{1}{4} + \frac{1009}{2016} > \frac{1}{2}$. Thus the closest integer to $a+b$ is 503.

2) What is the smallest number of seats in a large auditorium that must be occupied in order to be certain that at least two people share the same first and last initials?

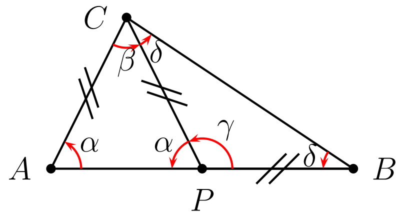
Answer: $26^2 + 1$ or 677.

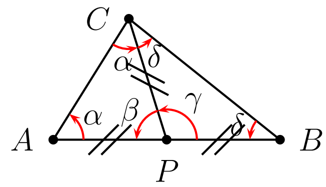
Solution: We shall use a "pigeon hole principle" argument. We are looking for identical first and last initials. There are 26 letters in the alphabet, so there are 26^2 different groups of people distinguished by their first and last initial. Thus, if we have $26^2 + 1$ occupied chairs there will be at least two people who will be in the same group based on their initials. Any number of chairs less than $26^2 + 1$ allows for a situation when all people will have different first and last initial since there are 26^2 different groups of people.

3) Using a line through one of its vertices a triangle is cut in two isosceles triangles. If the measure of the angle opposite one of the congruent sides in the first triangle is 40 degrees, what are the possible measurements (notice the plural!) for the angles opposite the congruent sides on the adjacent isosceles triangle?

Answer: $20^o, 80^o$ or 50^o.

Solution: There are two possible cases, see picture below, depending on the position of the isosceles triangles. The two mutually exclusive cases are: one of the smaller triangles has equal sides at the chosen vertex, or none of the smaller triangles has equal sides at the chosen vertex, see picture below.





It is enough to find the relation between the mentioned angles α and δ, using the notation on the pictures. Then, since we are given on of these two angles we can determine the other. In the first case displayed above, we have $\alpha = 2\delta$, while in the second case $\alpha + \delta = 90^o$. Hence the possible answers in the first case are $20^o, 80^o$, which are obtained when the given angle is $\alpha = 40^o$ or $\delta = 40^o$, respectively. In the second case there is only one answer since $90^o - 40^o = 50^o$.

4) Suppose that only eight tiles are left in the scrabble bag and the letters on the tiles spell CALCULUS. How many ways can you choose two tiles?

   Answer: 13 or 23 will be accepted.

   Solution: There are five different letters in the bag A, C, L, S, and U. There are three ways to select the double letters CC, LL and UU. The remaining selections use two distinct letters. Since there are five letters we have $\binom{5}{2} = 10$ ways of choosing two distinct letters. If order does matter, for example, if drawing AC is treated as different from CA, then there are $5 \cdot 4 = 20$ different ordered pairs. Hence, there are thirteen or twenty three ways to choose two tiles depending if order is taken into account.

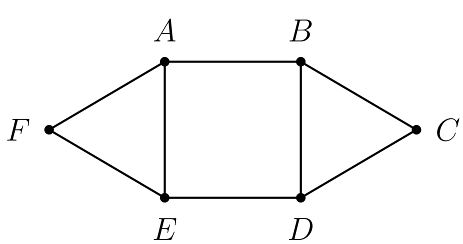
5) Suppose $0 \le a_i < n$ for $i = 0, 1, 2, \ldots, r$. The number $(a_r a_{r-1} \cdots a_1 a_0)_n$ represents the number $a_r n^r + \cdots a_1 n + a_0$ in base n. For example, $(102)_{13}$ is the base 13 representation of $1 \cdot 13^2 + 0 \cdot 13^1 + 2 \cdot 13^0 = 13^2 + 2 = 171$. In which bases n is $(11)_n$ a perfect square?

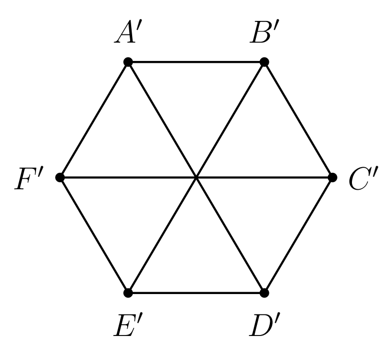
   Answer: All numbers which are one less than a perfect square or

$$n = \{m^2 - 1 \mid m = 2, 3, 4, \ldots\} = \{3, 8, 15, 24, \ldots\}.$$

   Solution: We want $(11)_n = n + 1 = m^2$, i.e., $n = m^2 - 1$. Note: n and m are positive integers, $m \ge 2$.

6) Using unit squares and equilateral triangles whose sides are of length one, you can form a convex hexagon with a unit square and two equilateral triangles whose sides are of length one or 6 equilateral triangles whose sides are length one as shown below.





How many unit squares and unit equilateral triangles can be used to construct a convex hexagon whose sides are all of length n?

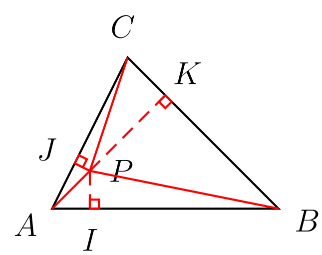
Answer: Any combination of $3n^2$ figures (n^2 squares and $2n^2$ triangles), or $6n^2$ figures (all triangles) for $n = 1, 2, 3, \dots$ will be accepted.

Solution: The above answers are obtained by using square pieces between the two "end" triangles for a total of $3n^2$ pieces in the first case shown on the picture above, or using only triangles, $6n^2$ of them, in the second case.

7) Let ΔABC be an equilateral triangle whose side is of length 1 inch. Let P be a point inside the triangle ΔABC. Find the sum of the distances of P to the sides of the triangle ΔABC.

Answer: $\frac{\sqrt{3}}{2}$.

Solution: Let d_1, d_2 and d_3 be the distances of P to the sides of the triangle. Connect P with each of the vertices of the given triangle, which splits the given triangle into three smaller triangles $\Delta PBC, \Delta PAC$ and ΔPAB.



The area S of the given triangle is the sum of the areas of the three small triangles. The formula for the area we want to use is that the area of a triangle is given by $\frac{1}{2}$ times the product of the length of any of its sides and the distance to it from the opposite vertex. Therefore, we have

$$S = \frac{1}{2}(d_1 + d_2 + d_3)$$

since the side of the triangle is 1. This shows that the sought sum $d_1 + d_2 + d_3$ is independent of the position of P inside the triangle, and equals twice the area. If we take P to be one of the vertices it follows $d_1 + d_2 + d_3 = h$, where h is the height of the equilateral triangle. This also follows from $S = \frac{1}{2}h$. In any case, we have to find h, which can be done in many ways. For example by the Pythagorean theorem we have

$$h^2 + (1/2)^2 = 1^2,$$

which gives

$$h = \frac{\sqrt{3}}{2}$$

8) Nathan just aced his math test and he is hoping that his parents will reward him for his performance. Nathan's parents decide that Nathan deserves a reward for his hard work; however, they like to add a little bit of chance to the reward. Nathan's parents have 5 crisp new 5 dollar bills and 5 crisp new 10 dollar bills. They tell Nathan that he has to divide the bills into two groups. Nathan's parents explain that after blindfolding Nathan they will place each group into a brown bag, after shuffling the bills. Then they will place one bag on the right hand side of a table and one on the left hand side of the table. He will choose one of the bags without examining them and then he will reach in and grab one of the bills. What is the highest probability that Nathan can achieve in picking a 10 dollar bill out of all possible groupings of the bills?

Answer: $\frac{13}{18}$.

Solution: If Nathan divides the bills so that one \$10 bill is alone and the remaining bills are all together in the other bag, he has a 50% chance of choosing the bag with the \$10 bill in it and 50% chance of choosing the other bag. If he chooses the bag with the only \$10 bill Nathan will get a \$10 bill with a probability $p_1 = 1$, while if he chooses the other bag, the probability of choosing a \$10 bill is $p_2 = 4/9$. Thus, the probability of choosing a \$10 bill with this distribution of the bills is

$$\frac{1}{2} \times p_1 + \frac{1}{2} \times p_2 = \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{4}{9} = \frac{13}{18}.$$

Any other combination will yield a lower probability since it will result in $p_1 < 1$ and $p_2 < 4/9$.

9) A boat is traveling against the flow of a river. Suppose the river is flowing at a constant speed and the boat maintains a constant speed with respect to the river while traveling in either direction along the river. At a certain moment of time a blow-up ball falls off the boat and starts floating down the river. 20 minutes after the ball fell into the water this was noticed and the boat reversed its direction and started going down the river chasing the ball. How long was the ball in the water before it was retrieved?

Answer: 40 (minutes).

Solution: Let T be the time after which it was noticed that the ball is missing. In the given set-up $T = 20$. However, let us consider a different set-up, based on the given one, where we reduce the speed of the river and that of the boat by the same quantity. We shall compute the time the ball was in the water in this more convenient set-up, but the answer will be enough to find the answer of the question in the given problem. Notice that everything depends on $v - c$ and $v + c$, where c is the speed of the river *relative to the banks of the river* and v is the speed of the the boat *relative to the river*. Thus, $v - c$ and $v + c$ are the speed of the boat *relative to the banks of the river* when going up-stream and down-stream, respectively. Notice that the differences $v - c$ and $v + c$ do not change if we subtract or add the same quantity from both v and c. So, let us reduce all velocities by the velocity of the river, i.e, in the new "world" the river will become stationary, which means that the boat will be traveling with the same speed going either up or down the river. In particular, if it takes T minutes to notice the missing ball, it will take another T minutes to get back to it (the ball is not moving!), hence the ball will be overall $2T$ minutes in the water. Now, if we go back to the given world we only need to notice that relative quantities will be preserved, hence it will be again $2T$ minutes until the ball is retrieved.

If you are not convinced by the above solution, then you can compute the time as follows. Let $\widehat{T}$ be the time the boat travels back down the river chasing the ball. So the ball is in the water $T + \widehat{T}$ minutes. During this time the ball travelled the distance $\left(T + \widehat{T}\right) c$, while the boat travelled a distance $(v - c)\, T$ up the river and $(v + c)\, \widehat{T}$ the river (all distances are measured with respect to the banks of the river). Since at the end, the boat and the ball are at the same place, from the place the ball was dropped, we have

$$\left(T + \widehat{T}\right) c = \widehat{T}\,(v + c) - T\,(v - c)\,.$$

Simplifying we find

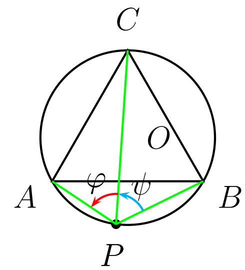
$$0 = \widehat{T}v - Tv,$$

hence $\widehat{T} = T = 20$.

10) Let ΔABC be an equilateral triangle. Find all points in the plane such that the distance from any such point to one of the vertices equals the sum of the distances to the remaining two.

Answer: The circumscribed circle.

Solution: Let a, b and c be the distances from P to the vertices A, B and C, respectively. Let m be the length of the sides of the equilateral triangle. Suppose P is a point on the arc of the circumscribed circle opposite the vertex C.



By the law of cosines we have

$$\begin{aligned} m^2 &= a^2 + c^2 - 2ac\cos\varphi = a^2 + c^2 - ac \\ m^2 &= b^2 + c^2 - 2bc\cos\psi = b^2 + c^2 - bc \\ m^2 &= a^2 + b^2 - 2ab\cos(\varphi + \psi) = a^2 + b^2 + ab, \end{aligned}$$

since $\varphi = \psi = 60^o$ and $\cos 60^o = -\cos 120^o = 1/2$. We want to show that $c = a + b$. Adding the first two of the above identities and multiplying the last one by 2 we obtain the equations

$$\begin{aligned} 2m^2 &= a^2 + b^2 + 2c^2 - c(a+b) \\ 2m^2 &= 2a^2 + 2b^2 + 2ab. \end{aligned}$$

Thus, we have

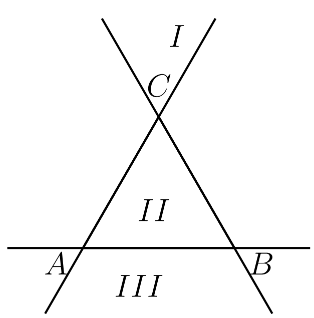
$$a^2 + b^2 + 2c^2 - c(a+b) = 2a^2 + 2b^2 + 2ab$$

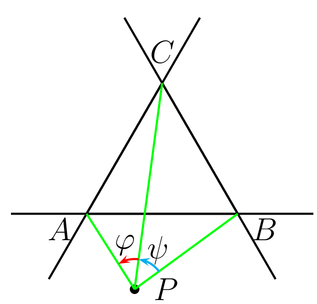
which implies

$$\begin{aligned} 0 &= (a+b)^2 + c(a+b) - 2c^2 = (a+b)^2 - c^2 + c(a+b) - c^2 \\ &= (a+b-c)(a+b+c) + c(a+b-c) = (a+b-c)(a+b+2c), \end{aligned}$$

hence $c = a + b$.

In order to see that no other points in the plane have this property we need a short argument. In addition, the argument will show how the above property can be found rather than just verified. The lines formed by the sides of the triangle ΔABC partition the plane into seven regions. Because of the symmetry (ΔABC is an equilateral triangle!) it is enough to find all point with the wanted property in the regions I, II and III below.





No P inside the triangle (region II) or region I (C is not included in it by definition) can have the desired property - this follows from the triangle inequality after a few calculations.

Let us consider the point in region III only. Those inside the circimscribed circle will have the property

$$60^o < \varphi \leq 90^o, 60^o < \psi \leq 90^o, 120^o < \varphi + \psi \leq 180^o,$$

while those outside the circimscribed circle will have the property

$$60^o > \varphi \geq 0^o, 60^o > \psi \geq 0^o, 120^o > \varphi + \psi \geq 0^o.$$

In other words we have either

$$\cos\varphi > 1/2, \cos\psi > 1/2, \cos(\varphi + \psi) < -1/2$$

or

$$\cos\varphi < 1/2, \cos\psi < 1/2, \cos(\varphi + \psi) > -1/2.$$

This can be re-written as the fact that exactly one of the following two lines of inequalities holds true:

$$\begin{aligned} 1 - 2\cos\varphi &< 0, 1 - 2\cos\psi < 0, 1 + 2\cos(\varphi + \psi) < 0 \\ 1 - 2\cos\varphi &> 0, 1 - 2\cos\psi > 0, 1 + 2\cos(\varphi + \psi) > 0. \end{aligned}$$

If we use the law of cosines as we did earlier we find

$$a^2 + b^2 + 2c^2 - c(a\cos\varphi + b\cos\psi) = 2a^2 + 2b^2 + 2ab\cos(\varphi + \psi),$$

which taking into account $c = a + b$, gves

$$\begin{aligned} &a^2(1 - 2\cos\varphi) + b^2(1 - 2\cos\psi) \\ &\qquad + 2ab\left[(1 + 2\cos(\varphi + \psi)) + \frac{1}{2}(1 - 2\cos\varphi) + \frac{1}{2}(1 - 2\cos\psi)\right] = 0. \end{aligned}$$

From the inequalities we derived it is clear that the right-hand side is either < 0 or > 0. In fact, we see that in order to be zero we must have

$$\varphi = 60^o, \psi = 60^o, \varphi + \psi = 120^o.$$

This proves again that the points on the circle have the desired property. We note explicitly that if P is on the arc oposite B, then $PA + PC = PB$,while if P is on the arc oposite A, then $PB + PC = PA$.