UNM–PNM STATEWIDE MATHEMATICS CONTEST XL

SOLUTIONS TO SECOND ROUND PROBLEMS

Any one who desires more details to any of the solutions below should contact me at

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1. You turn on a calculator and the screen reads '0'. The calculator can only display numbers smaller than 1×10^{100}. When you push the exponential button e^x the calculator computes and displays the exponential of whatever is on the calculator screen and similarly when you push the natural logarithm button $\ln x$ the calculator computes and displays the natural logarithm of whatever is on the calculator screen.

You have a coin which you flip. Each time the coin comes up heads you push the exponential button e^x. Each time the coin comes up tails you push the natural logarithm button $\ln x$.

a. After 3 flips, what is the probability that the calculator reads *Error*?

b. After 7 flips, what is the probability that the calculator reads *Error*?

You may use on this problem the approximation $2.7 < e < 2.8$.

For part **a.** the only type of error possible is taking the natural log of 0. This can happen on the first toss (4 possibilities) or on the third toss (only one possibility, namely HTT) so the probability is $\frac{5}{8}$ that the calculator reads Error after three flips.

For part **b.** there are two types of errors which can occur, taking the natural log of zero or overflow. To see how many times one needs to exponentiate to get an overflow error we use the fact that $2 < e < 3$. Thus

$$\begin{array}{rcl} e^0 & = & 1, \\ 2 < & e^1 & < 3, \\ 4 < & e^e & < 27, \\ 16 < & e^{e^e} & < 3^{27}. \end{array}$$

Clearly $e^{3^{27}}$ will be an error while e^{16} will not be. It is also clear that $e^{e^{16}}$ will be an error and thus one needs a little better approximation of e to decide whether the error will occur on the 5th or on the 6th time. Using $e > 2.7$ we see that $e^2 > 7$ and $e^e > 7\sqrt{2.7} > 7(1.5) > 10$. Thus $e^{e^e} > 2^{10} > 1000$ and so the overflow error definitely occurs on the 5th iteration of exponentiation. There are 8 total ways to obtain an overflow error, four where the first five tosses are heads, and four where there is one out of the seven which is tails occurring on the 2nd, 3rd, 4th, or 5th toss.

For the $\ln(0)$ error, this can only occur on the first, third, fifth, or seventh tosses and there are 64, 16, 8, 5 respective ways in which this can happen. Adding it all up, the probability of an error is 101/128.

2. Show that for any integer $n \geq 2$

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

is *not* a whole number. What about

$$1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n+1}?$$

Let M be the least common multiple of the denominators $2, \ldots, n$ and write $M = 2^r s$ where s is odd. Then $2^r \leq n < 2^{r+1}$ by the definition of r. We have

$$\frac{1}{2^r} = \frac{s}{M}$$

while for $1 \leq k \leq n$ *different* from 2^r we see that the largest power of 2 dividing k is at most $r-1$ and thus

$$\frac{1}{k} = \frac{a_k}{M}$$

where a_k is *even*. In particular the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

is equal to an odd number over M and hence is not a whole number.

The second part is similar except that one looks at powers of 3 instead of powers of 2. In particular, suppose

$$3^r \leq 2n + 1 < 3^{r+1}.$$

There is only one fraction $\frac{1}{k}$ in our sum with denominator divisible by 3^r, namely $\frac{1}{3^r}$, because the only other fraction with denominator less than 3^{r+1} divisible by 3^r is $\frac{1}{2 \cdot 3^r}$ which has an even denominator. Thus the sum

$$1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n+1}$$

has, when put over the least common multiple of the denominators, numerator which is not divisible by 3, and thus it cannot be a whole number.

3. The fraction $\frac{1}{6} = 0.1\overline{6}$ repeats after the *second* decimal place while the fraction $\frac{1}{13} = 0.\overline{076923}$ repeats after the *sixth* decimal place. Find when the decimals of the following fractions repeat:

a. $\frac{1}{28}$,

b. $\frac{1}{2008}$.

Calculation shows that

$$\frac{1}{28} = 0.03\overline{571428}$$

and the answer is thus eight. The important point to note here is that the repeating part of the decimal is the decimal for $\frac{4}{7}$. The reason for this is the following. We have $\frac{100}{28} = 3\frac{4}{7}$. But multiplication by 100 just moves the decimal two places to the right. For part **b.** note that $2008 = 8 \cdot 251$. Thus the decimal expansion of $\frac{1}{2008}$ will repeat three places *after* $\frac{1000}{2008} = \frac{125}{251}$. Note that 251 is a prime number. For a prime number p all fractions $\frac{a}{p}$ for $1 \leq a \leq p-1$ repeat/terminate after the *same* number of places (this is not obvious and is worth thinking about carefully!). Moreover, in the repeating case, the *entire decimal* repeats, as opposed to the case of $\frac{1}{6}$ where only the 6 repeats. Thus we will look at $\frac{1}{251}$ instead of $\frac{125}{251}$. The next important point (this is a form of Fermat's little theorem which follows from the fact that the decimals $\frac{a}{p}$ all repeat/terminate after the same number of places) is that the decimal of $\frac{1}{251}$ definitely repeats after 250 places but it might repeat earlier. It must, however, repeat after a *divisor* of 250 places (why?) so the possibilities are 1,2,5,10,25,50,125,250. Repeating after k places would mean that $10^k - 1$ is divisible by 251. Doing some modular arithmetic one finds that the decimal of $\frac{1}{251}$ repeats after 50 places and so that of $\frac{1}{2008}$ repeats after 53 places. To see that the calculations are not all that bad, note that $10^3 = 1000$ leaves a remainder of -4 when divided by 251 and so 10^{12} leaves a remainder of $(-4)^4 = 256$ which is the same as a remainder of 5. Hence 10^{24} leaves a remainder of 25 and 10^{25} a remainder of 250 which is the same as -1. In particular 10^{50} leaves a remainder of one when divided by 251. Since neither 10^2 nor 10^{10} leave a remainder of 1 (we already know that $10^1, 10^5$, and 10^{25} do not), 50 is the smallest possibility.

4.

a. Suppose ABC is a triangle and that the angle at vertex B is a right angle. Let P be the point on $\overline{AC}$ so that $\overline{BP}$ is perpendicular to $\overline{AC}$. Suppose $\overline{AP}$ has length a and $\overline{PC}$ has length 1. What is the length of $\overline{BP}$?

b. Suppose you are given a triangle T (*not* necessarily the triangle from part **a.**), a straightedge, a compass, and a line segment of unit length. Is it possible to construct a square S with the same area as T? If so, describe how *in detail* and if not prove that it is not possible.

The angles ABC, BPC, and APB are all right angles by hypothesis. Because they all have the same three angles, the three triangles ABC, BPC, and APB are similar. From the similarity of BPC and APB we deduce

$$\frac{\overline{BP}}{\overline{AP}} = \frac{\overline{PC}}{\overline{PB}},$$

where the bar denotes the length of the given segment. From this equality and the given information we see that $\overline{PB} = \sqrt{a}$.

Given a triangle ABC, label the vertices so that the angles BCA and BAC are both acute. There are then several basic steps to constructing a square S whose area is equal to the triangle ABC.

a. Draw perpendicular line from B to $\overline{AC}$, meeting $\overline{AC}$ at the point P.

b. Use the first part of problem 4 to construct segments of length $\sqrt{AC}$ and $\sqrt{BP}$. This requires a segment of unit length and the ability to draw a right angled triangle with a specified base.

c. Multiply $\sqrt{AC}$ and $\sqrt{BP}$. This also requires a construction and the use of similar triangles.

d. Construct a square of area one (which requires drawing perpendiculars). Its diagonal has length $\sqrt{2}$. Divide $\sqrt{AC} \cdot \sqrt{BP}$ by $\sqrt{2}$ (another geometric construction).

e. Use the length in part **d.** as your base for the square and then draw perpendicular lines to this base at the two endpoints ...

5. Consider the real numbers

$$\begin{aligned} x &= 0.1234567891011\ldots \\ e &= 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots. \end{aligned}$$

Thus x is obtained by listing, in order, all positive integers and, in the definition of e, $n!$ is the product of the first n whole numbers so that $2! = 2$, $3! = 6$, and so on.

a. Is x a rational number?

b. Is e a rational number?

For **a.** the answer is no. Any rational number $\frac{a}{b}$ has a terminating or a repeating decimal (because when doing long division of a by b there are only b possible different remainders and so either one gets a remainder of zero and the decimal terminates or two remainders repeat at which point the decimal repeats). The number x clearly does not terminate so one must show it does not repeat. Suppose x does have a repeating decimal of length n. If we go out far enough, we will find the number $1 \times 10^{n+1}$ which has $n+1$ consecutive zeroes. If x were a repeating decimal, the repeating part would therefore have to be all zeroes, that is x would be a terminating decimal which it definitely is not.

For **b.** suppose e is a rational number with denominator q. Then ae is a whole number whenever q divides a. In particular for any sufficiently large positive integer r the number $r!e$ is an integer. Using the definition of e, it follows that

$$\frac{1}{(r+1)} + \frac{1}{(r+1)(r+2)} + \ldots$$

is also a whole number whenever r is sufficiently large. This is impossible, however, as the displayed number is clearly positive and it is also less than one when r is large (why?).

6.

a. Find the polynomial $p(x)$ of degree three satisfying

$$\begin{aligned} p(-2) &= 0 \\ p(0) &= 6 \\ p(1) &= 3 \\ p(3) &= 45 \end{aligned}$$

b. Suppose d is a non–negative integer and suppose $a_1, \ldots a_{d+1}$ are *distinct* real numbers. Suppose $b_1, \ldots, b_{d+1}$ are (not necessarily distinct) real numbers. Show that there exists a *unique* polynomial $q(x)$ of degree *at most* d such that

$$q(a_i) = b_i \text{ for all } i.$$

For part **a.** the polynomial is $p(x) = 2x^3 - 5x + 6$. This can be found by plugging in the numbers –2, 0, 1, and 3 to the polynomial $p(x)$ and solving for the coefficients.

For part **b.** the desired polynomial is

$$P(x) = \sum_{i=1}^{d+1} b_i Q_i(x)$$

where

$$Q_i(x) = \frac{\prod_{j \neq i}(x - a_j)}{\prod_{j \neq i}(a_i - a_j)}.$$

The polynomials $Q_i(x)$ have degree d and have the property that they are zero at a_j for $j \neq i$ and equal to one at a_i. To see that $P(x)$ is unique, suppose that there were a distinct polynomial $Q(x)$ with the same properties. Then the polynomial $P(x) - Q(x)$

would vanish at $a_1, \ldots, a_{d+1}$ and this is impossible (because when a polynomial $f(x)$ vanishes at a this means that $f(x) = g(x)(x - a)$ where $g(x)$ has degree one less than f). The choice of $P(x)$ may look like something of a mystery but in fact it is *perfectly natural*. Indeed suppose $f(x)$ is a polynomial of degree at most r with $f(a_i) = b_i$ for $1 \leq i \leq r+1$. Then to get a polynomial $g(x)$ of degree at most$r+1$ with $g(a_i) = b_i$ for $1 \leq i \leq r+2$, there is no reason to spoil the nice properties $f(x)$ already has. To construct g you want to find a polynomial which vanishes at $a_1, \ldots, a_r$ (hence the numerator of $Q_i(x)$)) and then takes the right value at a_{r+1} (hence the denominator of $Q_i(x)$): adding this polynomial to $f(x)$ gives the desired $g(x)$.

7. Suppose T_1 and T_2 are two triangles with the *same* area.

a. Is it possible to cut T_1 into a finite number of smaller triangles which can be reassembled to make a rectangle R_1?

b. Is it possible to cut T_1 into a finite number of smaller triangles which can be reassembled to form T_2?

Label the triangle T as in problem 4. Let BP be the perpendicular from B to $\overline{AC}$ and let Q be the midpoint of $\overline{AP}$. Let R be the point of $\overline{AB}$ so that $\overline{RQ}$ is parallel to $\overline{BP}$. Then the triangle ARQ can be cut off and put back to complete a rectangle $BPQS$. The same construction applied to $\overline{PC}$ will make then turn T into a rectangle. A few extra cuts need to be made in order to decompose the new rectangle into triangles– in particular the quadrilateral RBPQ can be cut into two triangles and similarly with the other quadrilateral.

For part **b.** this is a very difficult problem, although the proof is perfectly elementary. The answer is yes and I will provide a reference for anyone interested in learning more.