UNM–PNM STATEWIDE MATHEMATICS CONTEST XL

Solutions

Anyone desiring details about a given problem is invited to contact me at

nakamaye@math.unm.edu

1. The obligatory questions about the current year:

a. What is the last digit of 3^{2007}?

b. What are the last *two* digits of 3^{2007}?

The last digit of 3^n follows a pattern: 3,9,7,1,3,9,7,1,.... The 2007^{th} element of this sequence is a 7.

For the last two digits, the pattern does not emerge until the 20^{th} element of the sequence and the last two digits are 87.

2. The real number x satisfying $x^3 = 2$ is called the cube root of 2 and is written $\sqrt[3]{2}$.

a. Find the whole number part and the first *two* decimal places of $\sqrt[3]{2}$.

b. What is $\sqrt[3]{2}$ rounded off to the nearest hundredth?

The answer to **a.** is 1.25 while the answer to **b.** is 1.26. No algorithm or formula is needed to solve this problem– enlightened guess and check is very efficient.

3. Suppose you have some dice that are six sided with the sides labeled one through six, and each side has an equal probability of occurring.

a. If you roll two dice what is the probability that the sum of the two dice is six?

b. If you roll three dice what is the probability that the sum of the three dice is eleven?

For **a.** there are six possible outcomes for each of the dice giving 36 total possibilities. The ones giving a sum of 6 are $(1,5)$, $(2,4)$, $(3,3)$, $(4,2)$, and $(5,1)$. Here the first number is the outcome of one die while the second number is the outcome for the other die. Thus the probability of a six is 5/36.

For **b.** one counts as in **a**. *depending* on the outcome of the first of the three dice. In other words if the first is a one then the other two have to add up to ten and there are 3 such possibilities. If the first is a two then the other two have to add up to nine and there are 4 such possibilities. Continuing in this way there are 27 ways to get a sum of 11 and there are 216 total outcomes giving an answer of 27/216 or 1/8.

4. What are the slopes of the two tangent lines from the origin in the plane to the circle of radius 1/2 centered at (1,1)?

The equation for the circle of radius one half centered at (1,1) is

$$(x-1)^2 + (y-1)^2 = 1/4.$$

The tangent lines are the values of a for which $y = ax$ meets the circle in *exactly* one point. Substituting in $y = ax$ into the equation of the circle, one wants exactly one real solution which means that the radical appearing in the quadratic formula must be $\sqrt{0}$. Solving this new quadratic equation gives

$$x = \frac{4 \pm \sqrt{7}}{3}.$$

5. Maria has a jar of pennies. When she arranges the pennies in rows of 15 she finds that there are seven left over. When she arranges them in rows of 19 she finds that there are eight left over. What is the *smallest* number of pennies that could be in the jar?

There are 217 pennies in the jar. This can be found by trial and error looking for the smallest positive a, namely $a = 11$, so that $19a + 8 - 7$ is divisible by 15.

6. Find the area of the triangle whose vertices, in the x-y plane, are $(2, 1), (8, 3), (6, 5)$.

The area is 8. The easiest way to see this is to draw a rectangle containing the triangle and subtracting the missing area.

7. The wheels on Heather's car are 23 inches in diameter. If she is traveling 60 miles per hour, how many times are the wheels rotating *per second* (rounded off to the nearest whole number)?

A car travelling 60 miles per hour is travelling $60 \cdot 5280 \cdot 12$ inches per hour and

$$\frac{60 \cdot 5280 \cdot 12}{3600} = 1056$$

inches per second. Dividing this by 23π, the circumference of the wheel, shows that the wheel is spinning almost 15 times per second. An approximation of π by 3.14 is enough to see this or one can use 3.1 and 3.2 as approximations.

8. What is the polynomial with leading coefficient 1 and integer coefficients which has $\sqrt{2+\sqrt{2}}$ as a root?

If $x = \sqrt{2+\sqrt{2}}$ then $x^2 = 2 + \sqrt{2}$ and so $x^2 - 2 = \sqrt{2}$ and $(x^2-2)^2 = 2$. Thus $x^4 - 4x^2 + 2 = 0$. There are also other polynomials with leading coefficient one satisfied by x but they all have degree larger than 4 and all are multiples of this one.