UNM–PNM STATEWIDE MATHEMATICS CONTEST XL

February 2, 2008 SECOND ROUND THREE HOURS

All answers must be fully justified!!

1. You turn on a calculator and the screen reads '0'. The calculator can only display numbers smaller than 1×10^{100}. When you push the exponential button e^x the calculator computes and displays the exponential of whatever is on the calculator screen and similarly when you push the natural logarithm button $\ln x$ the calculator computes and displays the natural logarithm of whatever is on the calculator screen.

You have a coin which you flip. Each time the coin comes up heads you push the exponential button e^x. Each time the coin comes up tails you push the natural logarithm button $\ln x$. You may use on this problem the fact that $2.7 < e < 2.8$.

a. After 3 flips, what is the probability that the calculator reads *Error*?

b. After 7 flips, what is the probability that the calculator reads *Error*?

2. Show that for any integer $n \geq 2$

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

is *not* a whole number. What about

$$1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n+1}?$$

3. The fraction $\frac{1}{6} = 0.1\overline{6}$ repeats after the *second* decimal place while the fraction $\frac{1}{13} = 0.\overline{076923}$ repeats after the *sixth* decimal place. Find when the decimals of the following fractions repeat:

a. $\frac{1}{28}$,

b. $\frac{1}{2008}$.

4.

a. Suppose ABC is a triangle and that the angle at vertex B is a right angle. Let P be the point on $\overline{AC}$ so that $\overline{BP}$ is perpendicular to $\overline{AC}$. Suppose $\overline{AP}$ has length a and $\overline{PC}$ has length 1. What is the length of $\overline{BP}$?

b. Suppose you are given a triangle T (*not* necessarily the triangle from part **a.**), a straightedge, a compass, and a line segment of unit length. Is it possible to construct a square S with the same area as T? If so, describe how *in detail* and if not prove that it is not possible.

5. Consider the real numbers

$$\begin{aligned} x &= 0.1234567891011\ldots \\ e &= 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots. \end{aligned}$$

Thus x is obtained by listing, in order, all positive integers and, in the definition of e, $n!$ is the product of the first n whole numbers so that $2! = 2$, $3! = 6$, and so on.

a. Is x a rational number?

b. Is e a rational number?

6.

a. Find the polynomial $p(x)$ of degree three satisfying

$$\begin{aligned} p(-2) &= 0 \\ p(0) &= 6 \\ p(1) &= 3 \\ p(3) &= 45 \end{aligned}$$

b. Suppose d is a non–negative integer and suppose $a_1, \ldots a_{d+1}$ are *distinct* real numbers. Suppose $b_1, \ldots, b_{d+1}$ are (not necessarily distinct) real numbers. Show that there exists a *unique* polynomial $q(x)$ of degree *at most* d such that

$$q(a_i) = b_i \text{ for all } i.$$

7. Suppose T_1 and T_2 are two triangles with the *same* area.

a. Is it possible to cut T_1 into a finite number of smaller triangles which can be reassembled to make a rectangle R_1?

b. Is it possible to cut T_1 into a finite number of smaller triangles which can be reassembled to form T_2?