1. Two people each have jobs. The first person works 40 hrs/week at x dollars per hour. The second person works 30 hrs/week at y dollars per hour. If they switch so that the first person works 30 hrs/week and the second person works 40 hrs/week, they will increase their combined income by \$100 per week. How much per hour does the second person make?

   Answer: 10

   Solution: Suppose the first person gets $\$x$ per hour, and the second gets $\$y$ per hour as their wages. Then

$$40x + 30y + 100 = 30x + 40y \Longrightarrow 10(y - x) = 100 \Longrightarrow y - x = 10.$$

2. Suppose a car going 65 mph is traveling North parallel to a train going 50mph. Once the front of the car is as far north as the back of the train, it takes another three minutes for the front of the car to be further North than the front of the train. How long is the train?

   Answer: 3/4 or 0.75 miles

   Solution: The difference in speeds is 15 mph, which means the distance between the front of the train and the front of the car decreases 1 mile every four minutes. Thus, in three minutes, the relative positions have changed by 3/4 of a mile, so the train is 3/4 of a mile long.

3. In how many ways can the number 2022 be written as the sum of consecutive integers?

   Answer: : 7 ways

   Solution: The sum of consecutive integers $= (number\ of\ numbers) \times (average\ of\ the\ first\ and\ the\ last\ numbers$
Since the average of two (the first and the last) integers is, itself, an integer, the average can be (only) an integer or an integer/2.
We start by noting that $2022 = 2 \cdot 3 \cdot 337$. This means, potentially, the number of numbers, paired with the corresponding averages, could be:
$(1, 2022), (2, 1011), (3, 674), (4, \frac{1011}{2}), (6, 337), (12, \frac{337}{2}), (337, 6), (674, 3), (1348, \frac{3}{2}),$
$(1011, 2), (2022, 1)$, and $(4044, \frac{1}{2})$.
For positive consecutive numbers, we cannot have an even number of numbers with an average that is an exact integer, so we can drop the $(2, 1011)$ and $(6, 337)$ cases.
[case 1] $(1, 2022)$ Since we require *consecutive* integers, we cannot have a single value of 2022 as a solution, but note that we can add the numbers $\{-2021, -2020, \ldots, 2020, 2021\}$ to the 2022, giving (including 0) 4044 numbers with an average of $\frac{1}{2}$.
[case 2] We can easily construct the $(3, 674)$ case as $673 + 674 + 675$. Moreover, this leads directly to
[case 3] We can add $\pm$ numbers -672,-671,...,0,1,...,672, which will all cancel, but will give us $-672 - 671 - \cdots - 1 + 0 + 1 + \cdots + 673 + 674 + 675 = 2022$ (the average is 3/2 and there are 1338 of them).
[case 4] Similarly, $(4, \frac{1011}{2})$ gives $504 + 505 + 506 + 507 = 2022$, and
[case 5] $-503 - 502 - \cdots + 502 + 503 + 504 + 505 + 506 + 507 = 2022$, for the $(1011, 2)$ pair. We

have remaining possible pairs of, $(12, \frac{337}{2}), (337, 6), (674, 3), (1011, 2), (2022, 1)$,
[case 6] For $(12, \frac{337}{2})$, we pick the six numbers on either side of 168.5, so $163+164+\cdots+173+174 = 2022$ (12 numbers). And, again, this leads directly to
[case 7] $-162 - 161 - \cdots + 173 + 174 = 2022$ (337 numbers, average is 6).
For the pair $(674, 3)$ (674 numbers, with an average of 3 for the fist and last numbers), we see that, for a set of negative numbers, zero, and the corresponding positive numbers, these add to 0, but produce an odd total number of numbers (because of zero). To get an average of 3 for the first and last requires adding six additional numbers, but this would give a total number of numbers as being odd. So this cannot happen.
Similarly for the $(2022, 1)$ case.
Thus, there are 7 cases that work.

4. Suppose $p > 0$. Find the smallest positive ϵ such that

$$\frac{p}{1+p} + \frac{p}{(1+p)(1+2p)} + \cdots + \frac{p}{(1+314158p)(1+314159p)} + \epsilon$$

is a whole number.

Note that

$$\frac{p}{(1+kp)(1+(k+1)p)} = \frac{1}{kp} - \frac{1}{1+(k+1)p}$$

This leads to a telescoping sum:

$$\frac{p}{1+p} + \frac{p}{(1+p)(1+2p)} + \cdots + \frac{p}{(1+2021p)(1+2022p)}$$
$$= \left(1 - \frac{1}{1+p}\right) + \left(\frac{1}{1+p} - \frac{1}{(1+p)(1+2p)}\right) + \cdots + \left(\frac{1}{1+2021p} - \frac{1}{1+2022p}\right)$$
$$= 1 - \frac{1}{1+2022p}$$

Thus,

$$\varepsilon = \frac{1}{1+2022p}$$

5. Find all the solutions to the equation

$$\sqrt[3]{25x(2x^2+9)} = 4x + \frac{3}{x}$$

Answer: $x = \pm\sqrt{3}, \pm i \cdot \sqrt{3/14}$

Solution: Cube both sides, then multiply both sides by x^3 to get rid of fractions. Then factor and use the quadratic formula.

$$\begin{aligned}
&\sqrt[3]{25x(2x^2+9)} = 4x + \frac{3}{x} \\
&\Rightarrow 25x(2x^2+9) = 64x^3 + 3(4x)^2(3/x) + 3(4x)(3/x)^2 + (3/x)^3 \\
&\Rightarrow 50x^3 + 225x = 64x^3 + 144x + 108/x + 27/x^3 \\
&\Rightarrow 14x^6 - 81x^4 + 108x^2 + 27 = 0 \\
&\Rightarrow 14y^3 + -81y^2 + 108y + 27 = 0 \quad y = x^2 \\
&\Rightarrow (y-3)^2(14y+3) = 0 \\
&\Rightarrow (x^2-3)^2(14x^2+3) = 0
\end{aligned}$$

The solutions are the roots of $x^2 - 3 = 0$ and $14x^2 + 3 = 0$.

6. Consider words consisting of letters from the alphabet $\{a, c, g, t\}$. How many words of length 8 are there where the first and letters are both a, and no two consecutive letters are the same?

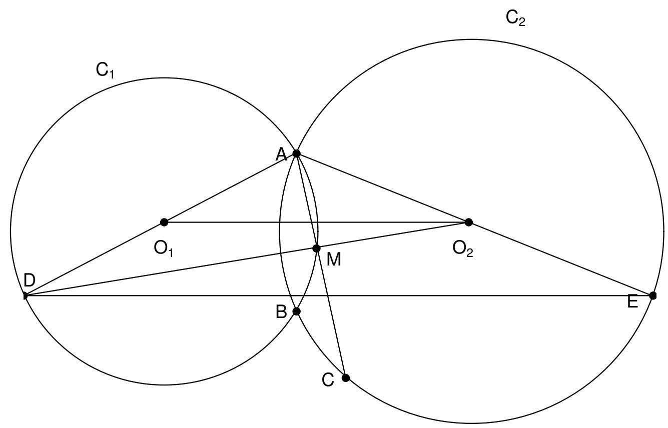
Answer: 546

Solution: The answer is 546. Let w_n be the number of words of length n. Note that $w_1 = 1$ for the words a and $w_2 = 0$ since aa has consecutive letters that are identical. $w_3 = 3$ with the words aca, aga, ata. We can set up a recursion for higher values of n. Suppose a word has an a in position $n-2$. Then after the a, one can append either ca, ga, or ta. Thus this contributes $3w_{n-2}$ new words. Suppose that a does not occur in position $n-2$. Then you have a sequence with $n-2$ letters. Adding an a at the end would give a sequence with $n-1$ letters that satisfy the constraints, which can happen in w_{n-1} ways. To get to a sequence of length n, replace the final a with xa where x is one of the non-a letters not in position $n-2$. Thus there are two choices. This leads to the recursion

$$w_n = 2w_{n-1} + 3w_{n-2}$$

The value for w_8 can be found by solving the recursion or just computing the first few terms. The answer is 546.

7. Circle C_1 has center O_1 and radius 1. Circle C_2 has center O_2 and radius $\sqrt{2}$. The circles intersect at points A and B (see diagram). Let AC be the chord of C_2 that is bisected by C_1. Find the length of AC given that O_1 and O_2 are 2 units apart.



Answer: $\sqrt{7/2}$.

Solution: Draw radii O_1A and O_2A. Extend these radii to diameters which intersect C_1 and C_2, respectively, at new points D and E. Here, the line DE necessarily goes through an intersection of the two circles, here point B.

The segment O_1O_2 joings the midpoints of the sides AD and AE and is therefore half the length of DE. Thus DE has length 4, AE has length $2\sqrt{2}$, and AD has length 2.

Letting M be the point of intersection of C_1 and DO_2, then anglle AMD is a right angle. Then M is the midpoint of chord AC, and $AC = 2AM$. Now the area of $\triangle ADO_2$ is $DO_2 \cdot AM/2$. Thus

$$AM = \frac{2\triangle ADO_2}{DO_2}$$

Also, since DO_2 is a meian of $\triangle ADE$,

$$AM = \frac{\triangle ADE}{DO_2}$$

The perimater of $\triangle ADE$ is $2 + 4 + 2\sqrt{2} = 6 + 2\sqrt{2}$, so half the perimeter is $3 + \sqrt{2}$. By Heron's formula,

$$\triangle ADE = \sqrt{(3+\sqrt{2})(1+\sqrt{2})(3-\sqrt{2})(-1+\sqrt{2})} = \sqrt{7}$$

By the Law of Cosines applied to $\triangle ADE$,

$$cosA = \frac{AD^2 + AE^2 - DE^2}{2AD \cdot AE} = -\frac{1}{\sqrt{8}}$$

Applying the Law of Cosines to $\triangle ADO_2$,

$$DO_2^2 = AD^2 + AO^2 - 2AD \cdot AO_2 cosA = 8$$

Thus,

$$AM = \sqrt{7}/\sqrt{8} \Rightarrow AC = \sqrt{7/2}$$

8. How many whole numbers between 1 and 2022 (inclusive) are perfect squares?

Answer: 44

Solution: Here $40^2 = 1600$, $45^2 = 2025$, which is slightly too high. $44^2 = 1936$, so 44 is the largest integer n such that $n^2 \leq 2022$.

9. Suppose A, B, C are points in a plane and $\angle ACB = \theta$. Describe the set of all the points X satisfying $|AX|^2 + |AB|^2 = |AC|^2$

Solution: (We assume that X is in the same plane as the points A, B, and C.) We have $|AX|^2 = |AC|^2 - |AB|^2$. If $|AB| > |AC|$, there are no solutions. If $|AB| = |AC|$, then $|AX| = 0$, and X is the single point A. Otherwise, if $|AC|^2 - |AB|^2 = r^2 > 0$, the set of solution points X is just the circle of radius r centered at A.

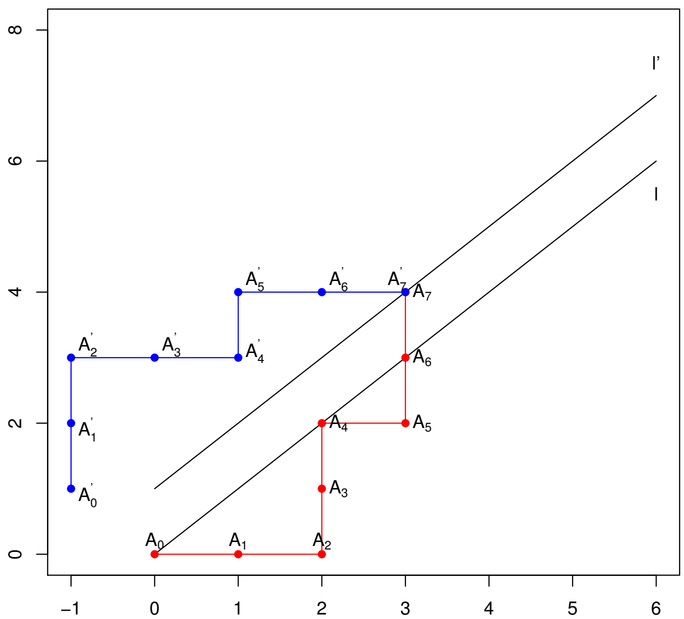
10. In a long line of people waiting to buy the latest edition of the popular magazine *Shiprock Mathematics Enthusiast*, each person either has a \$10 bill or a \$5 bill. If there are n people with \$5 bills and m people with \$10 bills, what is the probability the cashier will never run out of change if the cost of the magazine is \$5?

Answer: $\frac{n-m+1}{n+1}$.

Solution: (See plot below). Denote the first customer by a horizontal unit segment if they have \$5 and a vertical unit segment if they have \$10. Arrangements without enough change correspond to paths from $(0,0)$ to (n,n) which cross the line ℓ, or equivalently, have a vertex on ℓ'.

Suppose $A_0A_1 \cdots A_{n+m}$ is one such path and A_k is the first vertex on ℓ'. Then $A_0A_1 \cdots A_{n+m} \to A'_0 \cdots A'_{k-1}A'_k \cdots A_{n+m}$ is a bijection between all such paths and all "staircase" paths from $(-1,1)$ to (n,m). Since each such paath is fully specified by the positions of the $m-1$ vertical segments, the number of such maths is $\binom{n+m}{m-n}$. Similarly, the number of possible paths from $(0,0)$ to (n,m) is $\binom{n+m}{m}$. Thus,

$$p = 1 - \frac{\binom{n+m}{n-m}}{\binom{n+m}{m}} = \frac{n-m+1}{n+1}$$



l'
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A'_0
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