UNM – PNM STATEWIDE MATHEMATICS CONTEST LIII

February 6, 2020 Second Round Three Hours

1. Mr. Candelaria has three kids. They were all born in January but in different years. Mr. Candelaria's age is a multiple of the sum of kids' age, and also a multiple of the product of kids' age. The youngest kid's age is not less than 2 years, while Mr. Candelaria's age is not greater than 50 years.

   (a) What are the possible ages of kids? (Note: age must be an integer).
   (b) What is the probability that kids' birthdays are all on Saturday in 2021? (Note: The 1^{st} of January 2021 was a Friday, and January has a total of 31 days).

   Answer:
   (a) (2, 3, 5) or (2, 4, 6).
   (b) $(5/31)^3$

   Solution:
   (a) Given the youngest kid is 2 years old, since three kids were born in different years, the possible ages should be considered from 2, 3, and 4. The sum is 9 and product is 24, such that the least common multiple (LCM) is 72. This is not right because dad is not more than 50 years old. Similarly, we can list possible age combinations as following.

Age Combination	Sum	Product	LCM	Yes/No
2, 3, 4	9	24	72>50	No
2, 3, 5	10	30	30	Yes
2, 3, 6	11	36	>50	No
2, 3, 7	12	42	>50	No
2, 3, 8	13	48	>50	No
2, 4, 5	11	40	>50	No
2, 4, 6	12	48	48	Yes
2, 4, 7	13	56>50	>50	No
2, 5, 6	13	60>50	>50	No
3, 4, 5	12	60>50	>50	No

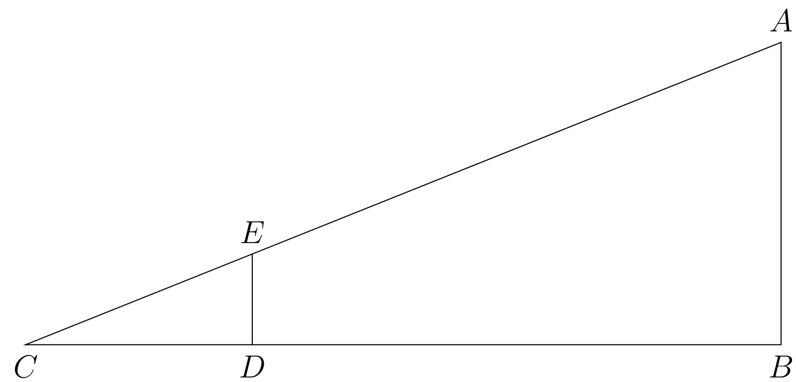
   (b) Since January 1st of 2021 is a Friday, then there are 5 Saturdays in January 2021 (Jan 2nd, 9th, 16th, 23rd, and 30th). Total number of days in January 2021 is 31. So the probability that one kid's birthday is on Saturday in 2021 is 5/31, and the probability of three kids' birthdays are all on Saturday is (5/31)(5/31)(5/31).

2. Sophia, a bright high school student, decided to experimentally estimate the height of the highest point on the Sandia Mountains. On a clear morning, while looking at the Sandias, Sophia could see the sun peeking above the peak. Sophia's height is h meters, and, with the sun still in the same place, she measured her own shadow on the ground to be s meters.

   Sophia was wearing a helmet with a laser mounted on top. After turning on the laser, the emitted beam reached the peak of the Sandias in t seconds, as measured by an accurate experimental setup she had previously devised for a science fair. After taking into account her experimental findings and assuming the laser beam traveled at a speed of c million meters per second, what height did Sophia estimate for the peak of the Sandias?

Answer: $\left(10^6 ct + \sqrt{s^2 + h^2}\right) \dfrac{h}{s}$ (meters).

Solution:



Let point A (see figure above) be the peak of Sandia Crest. AB is the height of Sandia Crest peak. ED is Sophia's height (h meters), and CD is Sophia's shadow (s meters).

$$EA = \text{ speed of light } \times \text{ time } = 10^6 ct \text{ (meters)}$$
$$CE = \sqrt{CD^2 + DE^2} = \sqrt{s^2 + h^2} \text{ (meters)}$$

Since $\triangle CDE$ is similar to $\triangle CBA$,

$$\frac{CE}{CA} = \frac{DE}{BA} \implies BA = \frac{CA \times DE}{CE} = \frac{(CE + AE) \times DE}{CE}$$

So $BA = \dfrac{(CE + AE) \times DE}{CE} = \left(10^6 ct + \sqrt{s^2 + h^2}\right) \dfrac{h}{s}$ (meters).

3. A computer scientist is writing an iPhone app in which she needs to evaluate the following expression:

$$\log \sum_{i=1}^{2021} e^{x_i},$$

where x_i are real numbers. However, the computer scientist notes that evaluating e^{x_i} for x_i larger than 88 results in the app crashing. How could she reformulate the above expression so that the new expression is mathematically the same, but computing it doesn't crash the app?

Answer: $a + \sum_{i=1}^{2021} e^{x_i - a}$

Solution:
Let $a = \max\{x_i\}$. Rewrite as

$$\log \sum_{i=1}^{2021} e^{x_i} = \log \sum_{i=1}^{2021} e^a e^{x_i - a} = \log e^a \sum_{i=1}^{2021} e^{x_i - a} = a + \sum_{i=1}^{2021} e^{x_i - a}$$

Note that $x_i - a < 88$.

4. How many distinct squares can be drawn on a grid 11 cells wide and 13 cells long? (Note: The length of each side of the square has to be a positive multiple of one cell width, and each corner of a square must be at some grid point).

Answer: 638 squares.

Solution: Consider the more general $n \times m$ case, and assume $n \leq m$. Since the grid is rectangular, the maximum size of a square is $n \times n$. For a $k \times k$ square, there are

$$(n-k+1)(m-k+1)$$

places to put the bottom left corner of the square on the grid. Therefore, the number of squares is

$$\sum_{k=1}^{n}(n-k+1)(m-k+1) = \frac{1}{6}n(n+1)(3m-n+1).$$

Plugging $n = 11$ and $m = 13$ into the above formula results in 638.

Note: The above calculation was all that was necessary for full credit. However, there are also such squares with sides not parallel to the sides of the rectangle. Each such rotated square may be embedded inside a square parallel to the edges of the rectangle, such that its vertices lie on the sides of the large square. The side of the smaller square then forms the hypotenuse of a triangle with integer height, width, and hypothenuse. Given the size of the rectangle and the possible choices of natural numbers a, b, c such that $a^2 + b^2 = c^2$, we find that the only rotated squares satisfying the conditions given in the problem are the two rotated squares embedded in each 7×7 square with sides parallel to the rectangle's side, i.e. the actual answer to the problem is $638 + 2(11-7+1)(13-7+1) = 708$.

5. For the set of positive real numbers $X = \{x_1, \ldots, x_n\}$, let

$$s_k = x_1^k + \cdots + x_n^k,$$

and p_k be the sum of all their possible products taken k at a time. Prove that

$$(n-1)!s_k \geq k!(n-k)!p_k.$$

(Example of a special case: Let $X = \{1, 2, 4\}$, and therefore $n = 3$. For $k = 2$, $s_k = 1^2+2^2+4^2 = 21$ and $p_k = (1)(2) + (1)(4) + (2)(4) = 14$. In this case, $(3-1)!21 = 42 \geq (2!)(3-2)!14 = 28$.)

Solution: Denote by $a_1, \ldots, a_k$ any k elements from the set. By the inequality of arithmetic and geometric means,

$$\frac{a_1^k + \cdots + a_k^k}{k} \geq \sqrt[k]{a_1^k \cdots a_k^k} = a_1 \cdots a_k.$$

For each x_i, there are exactly $\binom{n-1}{k-1}$ terms in p_k containing x_i. Therefore, by summing the above inequality over all k-long combinations from X, we get

$$\frac{\binom{n-1}{k-1} s_k}{k} \geq p_k,$$

from which

$$(n-1)!s_k \geq k(k-1)!(n-k)!p_k = k!(n-k)!p_k$$

immediately follows.

The counting part can be seen more explicitly as follows: Denote by $X_j = \{n_1^{(j)}, \ldots, n_k^{(j)}\}$ the jth k-long combination from $\{1, \ldots, n\}$, let $N = \binom{n}{k}$ the total number of such combinations, and $I_{X_j}(i)$ the function which equals 1 if $i \in X_j$ and 0 otherwise. Then:

$$\begin{aligned} p_k &= \sum_{j=1}^{N} x_{n_1^{(j)}} \cdots x_{n_k^{(j)}} \leq \sum_{j=1}^{N} \frac{x_{n_1^{(j)}}^k + \cdots + x_{n_k^{(j)}}^k}{k} \\ &= \frac{1}{k} \sum_{j=1}^{N} \sum_{i=1}^{n} x_i^k I_{X_j}(i) = \frac{1}{k} \sum_{i=1}^{n} x_i^k \sum_{j=1}^{N} I_{X_j}(i) \\ &= \frac{1}{k} \sum_{i=1}^{n} \binom{n-1}{k-1} x_i^k = \frac{1}{k} \binom{n-1}{k-1} s_k. \end{aligned}$$

6. The integer part of a real number x is the greatest integer less than or equal to x, and is denoted by $\lfloor x \rfloor$. For example, $\lfloor 4.5 \rfloor = 4$ and $\lfloor -4.5 \rfloor = -5$. Show that all non-negative solutions to the equation

$$x = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x}{5} \right\rfloor + \left\lfloor \frac{x}{6} \right\rfloor$$

are $x = 0, 4, 5$.

Solution: Note that

$$\begin{aligned} \left\lfloor \frac{x}{2} \right\rfloor &\leq \frac{x}{2} < \lfloor x \rfloor 2 + 1, \\ \left\lfloor \frac{x}{3} \right\rfloor &\leq \frac{x}{3} < \lfloor x \rfloor 3 + 1, \\ \left\lfloor \frac{x}{6} \right\rfloor &\leq \frac{x}{6} < \lfloor x \rfloor 6 + 1. \end{aligned}$$

Summing the three inequalities we get

$$\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{6} \right\rfloor \leq x < \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{6} \right\rfloor + 3.$$

By the given equality,

$$x - \left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x}{5} \right\rfloor = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{6} \right\rfloor,$$

and so

$$0 \leq \left\lfloor \frac{x}{4} \right\rfloor + \left\lfloor \frac{x}{5} \right\rfloor < 3.$$

Since $\left\lfloor \frac{x}{5} \right\rfloor \leq \left\lfloor \frac{x}{4} \right\rfloor$, we have that

$$0 \leq 2 \left\lfloor \frac{x}{5} \right\rfloor < 3,$$

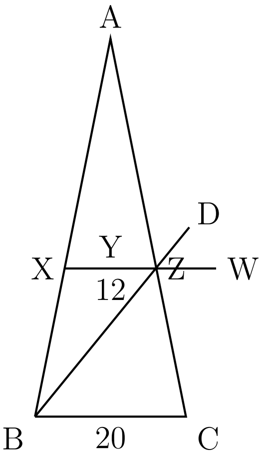
i.e. $0 \leq \lfloor x/5 \rfloor < 3/2$.

Since the floor function is always an integer, it must be that $\lfloor x/5 \rfloor = 0$ or $\lfloor x/5 \rfloor = 1$, and so $0 \leq x < 10$. Furthermore, since x is an integer (being the sum of evaluations of the floor function!), the only possible values for x are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, of which only $0, 4, 5$ satisfy the given equation.

7. A triangle ABC is such that the side BC has length 20 (units). Line XY is drawn parallel to BC such that X is on segment AB and Y is a point inside the triangle. The line XY (when extended) intersects CA at the point Z. Line BZ bisects angle YZC. If XZ has length 12 (units), then what is the length of AZ?

Answer: 30

Solution:



In the figure above we have angle WZD being equal to angle CBZ. By the problem statement, angle BZX is equal to angle BZC. But, angle WZD is equal to angle BZX. That is, we have angle BZC being equal to angle CBZ, and hence $\triangle BCZ$ is isoceles, with $BC = CZ = 20$. Now ABC and AXZ are similar triangles and hence,

$$\frac{AZ}{AC} = \frac{XZ}{BC}, \qquad \text{and so } \frac{AZ}{20 + AZ} = \frac{12}{20} \tag{1}$$

This leads to $20AZ = 240 + 12AZ$ or $AZ = 30$.

8. Find all polynomials $p(x)$ satisfying the identity $(x-1)p(x+1) = (x+2)p(x)$ for all real numbers x.

   Answer: $p(x) = c(x^3 - x)$, where c is an arbitrary constant.

   Solution: By substituting $x = 1$ and $x = -2$ into the identity, we get that

$$p(1) = p(-1) = 0.$$

Substituting $x = 0$ into the identity results in $-p(1) = 2p(0)$, from which we get that

$$p(0) = 0.$$

Therefore,

$$p(x) = x(x-1)(x+1)q(x) = x(x^2-1)q(x),$$

where q is some polynomial. Substituting this into the identity, we get that

$$x(x+2)(x^2-1)(q(x+1) - q(x)) = 0,$$

from which $q(x+1) = q(x)$ for all real values of x, which is possible if and only if $q(x) = c$ for some constant, and therefore

$$p(x) = c(x^3 - x),$$

where c is an arbitrary constant.

9. On New Year's Eve, yet another driver from Texas was seen speeding along I-40 near Santa Rosa. The license plate number is a four-digit number equal to the square of the sum of the two two-digit numbers formed by taking the first two digits and the last two digits of the license plate number. It's also known that the first digit of the license plate is not zero. What is the four-digit license plate number?

   Answer: $9801, 3025$ and 2025

Solution: Let the desired number be x. Let the number formed by its first two digits be a and the number formed by its last two digits be b. Then $x = 100a + b$. Then by the statement of the problem,

$$x = 100a + b = (a + b)^2 \tag{2}$$

or

$$99a = (a + b)^2 - (a + b) = (a + b)(a + b - 1) \tag{3}$$

This shows that the product of $(a + b)$ or $(a + b - 1)$ is divisible by 99. Hence we must have one of the following (for some integers k, m, n)
(a) $a + b = 99k$
(b) $a + b = 11m$ and $a + b - 1 = 9n$
(c) $a + b = 9m$ and $a + b - 1 = 11n$
(d) $a + b = 33m$ and $a + b - 1 = 3n$
(e) $a + b = 3m$ and $a + b - 1 = 33n$
(f) $a + b - 1 = 99k$
We investigate these one by one.

If $a+b = 99k$, then by Eq. (3) we have $a+b-1 = a/k$. Since a and b are two digit numbers, we must have $k \leq 2$. If $k = 1$ then $a + b = 99$ and $a + b - 1 = a/1$ and hence $a = 98$, $b = 1$. If $k = 2$ then we get $a = b = 99$. Thus $x = (98 + 1)^2 = 9801$.

If $a + b = 11m$ and $a + b - 1 = 9n$, then $99mn = 99a$ and $9n = 11m - 1$. That is, $11m - 1$ is divisible by 9 and hence it is easy to verify that $m = 5 \mathrm{mod} 9$. Thus, $m = 9t + 5$, and so $9n = 99t + 54, n = 11t + 6$. This gives. $a = mn = (9t + 5)(11t + 6) = 99t^2 + 109t + 30$. Since $a < 100$, we must have $t = 0$. This gives $a = 30$, $a + b = 11m = 55, b = 25$. Thus $x = (30 + 25)^2 = 3025$.

If $a + b = 9m$ and $a + b - 1 = 11n$, then again $mn = a$. Proceeding as before we arrive at $x = (20 + 25)^2 = 2025$.

Now $a + b = 33m$ and $a + b - 1 = 3n$ cannot happen since $a + b$ and $a + b - 1$ are relatively prime. Similarly $a + b = 3m$ and $a + b - 1 = 33n$ cannot happen.

The last case is $a+b-1 = 99k$, and hence $a+b = a/k$. The only possible value of k is $k = 1$, which leads to $a + b - 1 = 99$ and hence $a + b = 100$. Thus,

$$a = \frac{(a + b)(a + b - 1)}{99} = 100.$$

But since $a < 100$, this cannot happen.

Thus the only possible numbers are $9801, 3025$ and 2025.

10. In a round-robin tournament, n teams are put into $n/2$ pairs. Each pair of teams plays a game in round 1, and the winner moves to the next round. In round 2, the $n/2$ teams are paired, and the winners move on to round 3, which will have $n/4$ teams. This continues until the final round has two teams, and the winner of this final game wins the tournament. In a round-robin tournament with 2^n teams, there are n rounds.

    Suppose a round-robin sports tournament has 3 teams from Albuquerque and 13 teams from elsewhere in New Mexico, for a total of $16 = 2^4$ teams. Assume that each team is equally skilled so that each team has a 50% chance of winning each game; that all games are independent; and each game results in one team winning, so that there are no ties.

(a) Assuming that the initial pairings of the teams is done at random, and therefore all pairings are equally likely, what is the probability that one of the teams from Albuquerque wins the tournament?
(b) Assuming that the initial pairings of the teams is done at random, and therefore all pairings are equally likely, find the probability that in the first round, two of the teams from Albuquerque play each other.
(c) Instead of pairing teams at random, suppose that in round 1, two of the Albuquerque teams are paired, while the third Albuquerque team is paired with another team at random. This guarantees that at least one Albuquerque team wins and at least one Albuquerque team loses in the first round. Also suppose that if there are two Albuquerque teams available in round 2, they are planned to play each other. Thus in round three there are either 0 or 1 Albuquerque teams. Under this setting, find the probability that an Albuquerque team wins the tournament.

Answer:
(a) 3/16
(b) 1/5 (note that the previous solution read 1/40)
(c) 3/16

Solution:
(a) Each team has the same probability of 3/16.
(b) The probability that the kth pairing is an Albuquerque vs. Albuquerque pairing is

$$p = \frac{\binom{3}{2}}{\binom{16}{2}} = \frac{1}{40}.$$

Since there are eight pairings, the answer is $8p = 1/5$.
(c) Let the three Albuquerque teams be A_1, A_2, A_3. Assume that A_1 and A_2 play each other, and A_3 plays some other team. Then the required probability $P$is,

$$\begin{aligned} P &= \text{Probability some Albuquerque team wins given that } A_3 \text{ wins } \times A_3\text{wins} \\ &\quad + \text{Probability some Albuquerque team wins given that } A_3 \text{ loses} \times A_3\text{loses} \\ &= \frac{1}{4} \times \frac{1}{2} + \frac{1}{8} \times \frac{1}{2} = \frac{3}{16}. \end{aligned}$$