UNM - PNM STATEWIDE MATHEMATICS CONTEST XLIX

February 4, 2017 Second Round Three Hours

1. What are the last two digits of 2017^{2017}?

   Answer: 77. From Round 1, we know that $(2017)^4 \equiv 7^4 \equiv 1 \bmod 10$. Thus $7^{4k+1} \equiv 7 \bmod 10$ for all integers k. Note that $2017 = 4(504)+1$. So the units digit must be 7. To determine the digit in the 10's spot, it should help to find the smallest power k such that $(2017)^{4k} \equiv (17)^{4k} \equiv 1 \bmod 100$.

   Note that

$$17^4 \equiv 21 \bmod 100$$
$$17^{4\cdot 2} \equiv 41 \bmod 100$$
$$17^{4\cdot 3} \equiv 61 \bmod 100$$
$$17^{4\cdot 4} \equiv 81 \bmod 100$$
$$17^{4\cdot 5} \equiv 1 \bmod 100$$

   Thus $2017^{2017} \equiv 2017^{17} \bmod 100$. This will be precisely $17^{16} \cdot 17 \equiv 81 \cdot 17 \equiv 77 \bmod 100$

2. Suppose A, R, S, and T all denote distinct digits from 1-9. If $\sqrt{STARS} = SAT$, what are A, R, S, and T?

   Answer: $A = 3$, $T = 9$, $S = 1$ and $R = 2$. Since SAT is a 3 digit number and $SAT^2 = STARS$ is a 5 digit number then $1 \leq S \leq 3$. However since that digit is S and only $1^2 = 1$, $S = 1$. Now since S and T are distinct and the only other digit a with $a^2 \equiv 1 \bmod 10$ is $a = 9$, $T = 9$. Now to determine A. We know that SA^2 must begin with ST. Note that $15^2 = 225$, so $1 < A < 5$. We will look at $129^2 = 16641$ and $139^2 = 19321$ and $149^2 = 22201$. The only one that fits the bill is 139. So $A = 3$ and $R = 2$.

3. Let $f(x) = \dfrac{x}{x-1}$ and $g(x) = \dfrac{x}{3x-1}$.

   (a) Determine $f \circ g(x)$ and $g \circ f(x)$.

   (b) Denote $\underbrace{h \circ h \circ \cdot \circ h}_{n \text{ times}} := h^n$. Determine all the functions in the set

$$S = \{H \mid H = (g \circ f)^n \circ g \text{ or } H = (f \circ g)^n \circ f \text{ for some } n \text{ a whole number}\}.$$

   Answer: (a) $f \circ g(x) = \dfrac{x}{-2x+1}$ and $g \circ f(x) = \dfrac{x}{2x+1}$ (b) $S = \{\dfrac{x}{nx-1} \mid$ where n is an odd integer.$\}$

   Note that (a) is straightforward. First we need to determine $(f \circ g)^n$ we will find a formula using induction.

$$(f \circ g)^2 = \frac{\dfrac{x}{-2x+1}}{\dfrac{-2x}{-2x+1}+1} = \frac{x}{-4x+1}$$

   Suppose $(f \circ g)^n = \dfrac{x}{-2nx+1}$, we easily see that $(f \circ g)^{n+1} = \dfrac{x}{-2(n+1)x+1}$ as the computation above. So by induction $(f \circ g)^n = \dfrac{x}{-2nx+1}$.

   Similarly $(g \circ f)^n = \dfrac{x}{2nx+1}$. Now

$$(f \circ g)^n \circ f = \frac{x}{(-2n+1)x-1}$$

and

$$(g \circ f)^n \circ g = \frac{x}{(2n+3)x - 1}.$$

The $(f \circ g)^n \circ f$ give us the set

$$\left\{ \frac{x}{(-2n+1)x - 1} \mid n \geq 0 \right\}$$

and the $(g \circ f)^n \circ g$ give us the set

$$\left\{ \frac{x}{(2n+3)x - 1} \mid n \geq 0 \right\}.$$

So S is the union of these 2 sets and hence,

$$S = \left\{ \frac{x}{(2n+1)x - 1} \mid n \in \mathbb{Z} \right\}.$$

4. Find a second-degree polynomial with integer coefficients, $p(x) = ax^2 + bx + c$, such that $p(1), p(3), p(5)$, and $p(7)$ are perfect squares, but $p(2)$ is not.
Answer: A polynomial that satisfies the criteria is easily constructed by first centering it at $x = 4$, that is
$$p(x) = \hat{a}(x-4)^2 + \hat{c}.$$
Now we have two conditions: $n^2 = p(1) = 9\hat{a} + \hat{c}$ and $m^2 = p(3) = \hat{a} + \hat{c}$ that determines possible candidates that can then be checked against the condition that $p(2)$ is not a perfect square and the condition that the coefficients are integers. A few possible answers are $(n, m, \hat{a}, \hat{c}) = (0, 4, -2, 18), (1, 3, -1, 10), (2, 6, -4, 40), (3, 5, -2, 27), \ldots$.

5. Find all real triples (x, y, z) which are solutions to the system:

$$\begin{aligned} x^3 + x^2y + x^2z &= 40 \\ y^3 + y^2x + y^2z &= 90 \\ z^3 + z^2x + z^2y &= 250 \end{aligned}$$

Answer: There are 3 solutions $(2, 3, 5)$ or $(-\sqrt[3]{\frac{40}{3}}, \sqrt[3]{45}, \sqrt[3]{\frac{54}{3}})$ or $(\sqrt[3]{20}, -\frac{3\sqrt[3]{20}}{2}, \frac{5\sqrt[3]{20}}{2})$.
Note that the three equations imply that
$$\frac{40}{x^2} = \frac{90}{y^2} = \frac{250}{z^2}$$
implying that $y = \pm\frac{3x}{2}$ and $z = \pm\frac{5x}{2}$.
Case 1: $y = \frac{3x}{2}$ and $z = \frac{5x}{2}$. Then $x^2(x + \frac{3x}{2} + \frac{5x}{2}) = 40$ or $5x^3 = 40$ whose only real solution is $x = 2$. This gives the triple $(2, 3, 5)$.
Case 2: $y = -\frac{3x}{2}$ and $z = \frac{5x}{2}$. Then $x^2(x - \frac{3x}{2} + \frac{5x}{2}) = 40$ or $2x^3 = 40$ whose only real solution is $x = \sqrt[3]{20}$. This gives the triple $(\sqrt[3]{20}, -\frac{3\sqrt[3]{20}}{2}, \frac{5\sqrt[3]{20}}{2})$.
Case 3: $y = \frac{3x}{2}$ and $z = -\frac{5x}{2}$. Then $x^2(x + \frac{3x}{2} - \frac{5x}{2}) = 40$ or $0 = 40$ which implies there is no real solution for this case.
Case 4: Then $y = -\frac{3x}{2}$ and $z = -\frac{5x}{2}$. Then $x^2(x - \frac{3x}{2} - \frac{5x}{2}) = 40$ or $-3x^3 = 40$ whose only real solution is $x = -\sqrt[3]{\frac{40}{3}}$. This gives the triple $(-\sqrt[3]{\frac{40}{3}}, \sqrt[3]{45}, \sqrt[3]{\frac{54}{3}})$.

6. There are 12 stacks of 12 coins. Each of the coins in 11 of the 12 stacks weighs 10 grams each. Suppose the coins in the remaining stack each weigh 9.9 grams. You are given one time access to a precise digital scale. Devise a plan to weigh some coins in precisely one weighing to determine which pile has the lighter coins.

   Answer: On the digital scale, place one coin from the 1st pile, 2 from the 2nd, 3 from the 3rd, continuing in this fashion until you have placed 12 from the 12th pile. The scale will have $\sum_{i=1}^{12} i = \frac{12 \cdot 13}{2} = 78$ coins. If the pile weighs 779.9g, the first pile has the lighter coins. If the pile weighs 779.8g, the second pile has the lighter coins. And in general, if the pile weighs $780 - i(.1)$, the ith pile has the lighter coins.

7. Find a formula for $\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{n}{4k}$ for any natural number n.

   Answer: The formula is $\dfrac{2^n + (1+i)^n + (1-i)^n}{4}$.

   First recall from the binomial theorem that $\binom{n}{4k}$ is the coefficient of x^{4k} in the expansion of $(1+x)^n$. The sum above only includes the coefficients of the powers of x which are divisible by 4.

   Note that $1 = i^{4k} = (-1)^{4k} = (-i)^{4k}$. For any k, $1 + i^{4k} + (-1)^{4k} + (-i)^{4k} = 4$.

   However, because $a^{4k+1} = a$, for any k and $a = 1, i, -1, -i$, then

$$1 + i^{4k+1} + (-1)^{4k+1} + (-i)^{4k+1} = 1 + i - 1 - i = 0.$$

   Similarly, since $i^{4k+3} = -i$, $(-i)^{4k+3} = i$ and $(-1)^{4k+3} = -1$ for any k, then

$$1 + i^{4k+3} + (-1)^{4k+3} + (-i)^{4k+3} = 1 - i - 1 + i = 0.$$

   Lastly,

$$1 + i^{4k+2} + (-1)^{4k+2} + (-i)^{4k+2} = 1 - 1 + 1 - 1 = 0.$$

   Using the binomial theorem

$$(1+1)^n + (1+i)^n + (1+-1)^n + (1+-i)^n = \sum_{k=0}^{n} \binom{n}{k} + \sum_{k=0}^{n} \binom{n}{k} i^k + \sum_{k=0}^{n} \binom{n}{k} (-1)^k + \sum_{k=0}^{n} \binom{n}{k} (-i)^k.$$

   Using the relations above almost all of the binomial coefficients cancel and we obtain

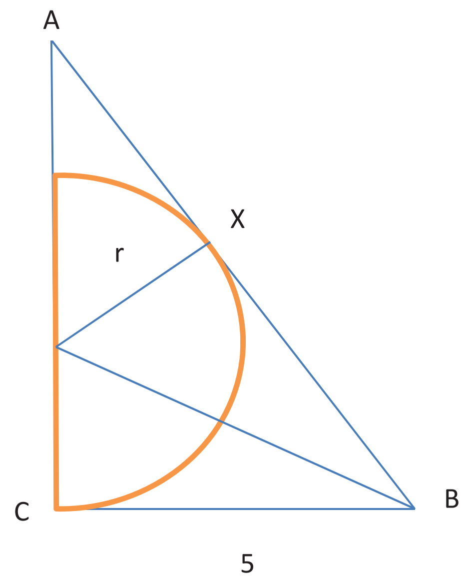
$$(1+1)^n + (1+i)^n + (1+-1)^n + (1+-i)^n = 4 \sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{n}{4k}.$$

   Hence, the formula is $\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{n}{4k} = \dfrac{2^n + (1+i)^n + (1-i)^n}{4}$.

8. Let ABC be a right triangle with right angle at C. Suppose $AC = 12$ and $BC = 5$ and CX is the diameter of a semicircle, where X lies on AC and the semicircle is tangent to side AB. Find the radius of the semicircle.

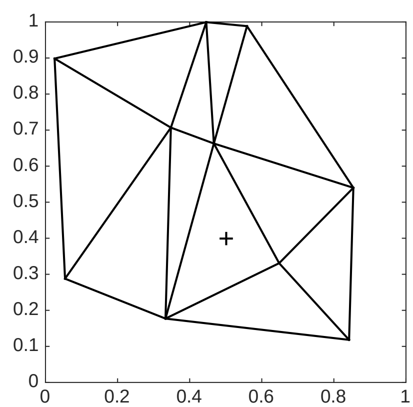
   Answer: 10/3.

   Consider the following triangle:



We can compute the area in two ways $A = 1/2 \cdot 5 \cdot 12 = 30$ or $A = 1/2 \cdot r \cdot 13 + 1/2 \cdot r \cdot 5 = 9r$. Setting the two areas equal we obtain $r = 10/3$.

9. Consider a triangulation (mesh) of a polygonal domain like the one in the figure below.
   (a) Given the vertices of a triangle, devise a strategy for determining if a given point is inside that triangle.
   (b) Will your strategy work for polygons with more than three sides?
   (c) After implementing your strategy in an optimally efficient computer code you find that the search for a problem with 100 triangles, on average, takes 10 seconds. You refine the triangulation by subdividing each of the triangles into smaller triangles by placing a new vertex at the center of gravity of each triangle. On average, how long will it take to find a point in the new mesh?



Answer: Various solutions are possible. For example, a simple algorithm is to walk around the polygon along one side at the time. If the point is always to the left (or always to the right) of the line in the plane that coincides with the current side, then the point is inside (for triangles this is done by dot-products).

Another option is to use the *Jordan Curve Theorem*: "Any continuous simple closed curve in the plane, separates the plane into two disjoint regions, the inside and the outside". This theorem implies that if one casts a ray from the point at hand in any direction and count the number of intersections of the ray and the boundary of the polygon. If the number of crossings is odd then the point is inside, see Figure **??**.

The complexity will typically be linear so the new search will, on average, take half a minute as the number of triangles have grown by a factor of three.

This problem is of significant practical importance in computer graphics. Additional information can be found at: https://en.wikipedia.org/wiki/Point_in_polygon. If one only consider triangles there are three techniques that are particularly popular: barycentric coordinate system, parametric equations system, check sides with dot product. These are described (along with explicit formulas and computer codes) in this blog post:

http://totologic.blogspot.fr/2014/01/accurate-point-in-triangle-test.html

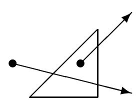


Figure 1. Ray casting.

10. Newton's method applied to the equation $f(x) = x^3 - x$ takes the form of the iteration

$$x_{n+1} = x_n - \frac{x_n^3 - x_n}{3x_n^2 - 1}, \quad n = 0, 1, 2, \ldots$$

(a) What are the roots of $f(x) = 0$?
(b) Study the behavior of the iteration when $x_0 > 1/\sqrt{3}$ to conclude that the sequence $\{x_0, x_1, \ldots\}$ approaches the same root as long as you choose $x_0 > 1/\sqrt{3}$. It may be helpful to start with the case $x_0 > 1$.
(c) Assume $-\alpha < x_0 < \alpha$. For what number α does the sequence always approach 0?
(d) For $x_0 \in (\alpha, 1/\sqrt{3})$ the sequence may approach either of the roots $\pm x^*$. Can you find an (implicit) expression that can be used to determine limits a_i and a_{i+1} such that if $x_0 \in (a_i, a_{i+1})$ then the sequence approaches $(-1)^i x^*$. Hint: $a_1 = 1/\sqrt{3}$, $a_i > a_{i+1}$ and a_i approaches $1/\sqrt{5}$ when i becomes large.

Answer:

Answer: (a) The roots are -1,0,1. (b) First consider $x_0 > 1$. Let $x_{n+1} = 1 + \varepsilon$ and $x_n = 1 + \delta$ with $\delta > 0$. The iteration gives $0 < \frac{\varepsilon}{\delta} < \frac{2}{3}$. Next consider $1/\sqrt{3} < x_0 < 1$. As the signs of the numerator and denominator in the rational part of the iteration does not change on the interval under consideration we find that $x_1 > 1$. Finally, $x_0 = 1$ produces $x_1 = 1$.

To answer (c), rewrite the iteration as $x_{n+1} = -\frac{2x_n^3}{1-3x_n^2}$, and note that for $0 \le x_0 < 1/\sqrt{3}$ the next iterate will be non-positive. Insisting that $-x_0 < x_1 \le 0$, so that x_1 will be closer to zero than x_0 gives the limiting case $x_1 = -x_0$, or $\alpha(1 - 3\alpha^2) = -2\alpha^3$, which has the solution $\alpha = 1/\sqrt{5}$.

Finally the implicit recurrence in (d) is obtained by running Newton backwards

$$a_i - \frac{a_i^3 - a_i}{3a_i^2 - 1} = -a_{i-1}, \quad a_1 = 1/\sqrt{3}, \ldots, a_\infty = 1/\sqrt{5}.$$