UNM - PNM STATEWIDE MATHEMATICS CONTEST XLIX

February 4, 2017 Second Round Three Hours

1. What are the last two digits of 2017^{2017}?

2. Suppose A, R, S, and T all denote distinct digits from 1-9. If $\sqrt{STARS} = SAT$, what are A, R, S, and T?

3. Let $f(x) = \dfrac{x}{x-1}$ and $g(x) = \dfrac{x}{3x-1}$.
   (a) Determine $f \circ g(x)$ and $g \circ f(x)$.
   (b) Denote $\underbrace{h \circ h \circ \cdot \circ h}_{n \text{ times}} := h^n$. Determine all the functions in the set

$$S = \{H \mid H = (g \circ f)^n \circ g \text{ or } H = (f \circ g)^n \circ f \text{ for some } n \text{ a whole number}\}.$$

4. Find a second-degree polynomial with integer coefficients, $p(x) = ax^2 + bx + c$, such that $p(1), p(3), p(5)$, and $p(7)$ are perfect squares, but $p(2)$ is not.

5. Find all real triples (x, y, z) which are solutions to the system:

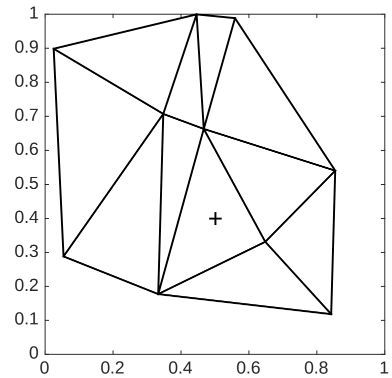
$$\begin{aligned} x^3 + x^2y + x^2z &= 40 \\ y^3 + y^2x + y^2z &= 90 \\ z^3 + z^2x + z^2y &= 250 \end{aligned}$$

6. There are 12 stacks of 12 coins. Each of the coins in 11 of the 12 stacks weighs 10 grams each. Suppose the coins in the remaining stack each weigh 9.9 grams. You are given one time access to a precise digital scale. Devise a plan to weigh some coins in precisely one weighing to determine which pile has the lighter coins.

7. Find a formula for $\sum_{k=0}^{\lfloor \frac{n}{4} \rfloor} \binom{n}{4k}$ for any natural number n.

8. Let ABC be a right triangle with right angle at C. Suppose $AC = 12$ and $BC = 5$ and CX is the diameter of a semicircle, where X lies on AC and the semicircle is tangent to side AB. Find the radius of the semicircle.

9. Consider a triangulation (mesh) of a polygonal domain like the one in the figure below.
   (a) Given the vertices of a triangle, devise a strategy for determining if a given point is inside that triangle.
   (b) Will your strategy work for polygons with more than three sides?
   (c) After implementing your strategy in an optimally efficient computer code you find that the search for a problem with 100 triangles, on average, takes 10 seconds. You refine the triangulation by subdividing each of the triangles into smaller triangles by placing a new vertex at the center of gravity of each triangle. On average, how long will it take to find a point in the new mesh?



10. Newton's method applied to the equation $f(x) = x^3 - x$ takes the form of the iteration

$$x_{n+1} = x_n - \frac{x_n^3 - x_n}{3x_n^2 - 1}, \quad n = 0, 1, 2, \ldots$$

   (a) What are the roots of $f(x) = 0$?
   (b) Study the behavior of the iteration when $x_0 > 1/\sqrt{3}$ to conclude that the sequence $\{x_0, x_1, \ldots\}$ approaches the same root as long as you choose $x_0 > 1/\sqrt{3}$. It may be helpful to start with the case $x_0 > 1$.
   (c) Assume $-\alpha < x_0 < \alpha$. For what number α does the sequence always approach 0?
   (d) For $x_0 \in (\alpha, 1/\sqrt{3})$ the sequence may approach either of the roots $\pm x^*$. Can you find an (implicit) expression that can be used to determine limits a_i and a_{i+1} such that if $x_0 \in (a_i, a_{i+1})$ then the sequence approaches $(-1)^i x^*$. Hint: $a_1 = 1/\sqrt{3}$, $a_i > a_{i+1}$ and a_i approaches $1/\sqrt{5}$ when i becomes large.