**UNM-PNM Statewide High School Mathematics Contest**
**Round-2, 10 February 2024, 14:00-17:30**

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**1.** Consider a sum of natural numbers such that each digit from 1 to 8 appears only once. For example, the numbers 81, 27, 4536 feature each digit from 1 to 8 only once and sum to $81 + 27 + 4536 = 4644$.

**(a)** Find such a sum which adds up to 243.

**(b)** Find the integer nearest 2024 which can be represented as such a sum.

**Solution.** **(a)** There are many possible solutions; here a few

$$213 + 4 + 5 + 6 + 7 + 8, \qquad 147 + 82 + 3 + 5 + 6, \qquad 81 + 62 + 53 + 47.$$

**(b)** As shown below, all such sums are divisible by 9; whence $2024 = 2^3 \cdot 11 \cdot 23$ cannot be represented as such a sum. However, $2025/9 = 225$ which suggests such a representation is possible for 2025. One such sum is $1587 + 432 + 6 = 2025$.

**Remarks.** We might have found any of the sums reported in **(a)** via experimentation, but let us think systematically. Clearly, there is no such sum of 8 summands, since $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$. Likewise, there is no such sum of 7 summands, since the largest such sum is $1+2+3+4+5+6+87 = 108$. What about a sum of 6 summands? There are several possibilities.

$$\begin{aligned}
213 + 4 + 5 + 6 + 7 + 8 &= 243\\
3 + 214 + 5 + 6 + 7 + 8 &= 243\\
3 + 4 + 215 + 6 + 7 + 8 &= 243\\
3 + 4 + 5 + 216 + 7 + 8 &= 243\\
3 + 4 + 5 + 6 + 217 + 8 &= 243\\
3 + 4 + 5 + 6 + 7 + 218 &= 243.
\end{aligned}$$

For a sum of 5 summands, consider

$$\begin{aligned}
182 + 43 + 5 + 6 + 7 &= 243\\
187 + 42 + 3 + 5 + 6 &= 243\\
186 + 47 + 2 + 3 + 5 &= 243\\
185 + 46 + 7 + 2 + 3 &= 243\\
183 + 45 + 6 + 7 + 2 &= 243\\
142 + 83 + 5 + 6 + 7 &= 243\\
147 + 82 + 3 + 5 + 6 &= 243\\
146 + 87 + 2 + 3 + 5 &= 243\\
145 + 86 + 7 + 2 + 3 &= 243\\
143 + 85 + 6 + 7 + 2 &= 243.
\end{aligned}$$

For a sum of 4 summands, here are some possibilities.

$$\begin{aligned}
123 + 65 + 47 + 8 &= 243\\
128 + 63 + 45 + 7 &= 243\\
127 + 68 + 43 + 5 &= 243\\
125 + 67 + 48 + 3 &= 243,
\end{aligned}$$

and

$$\begin{aligned}81+62+53+47&=243\\87+61+52+43&=243\\83+67+51+42&=243\\82+63+57+41&=243.\end{aligned}$$

There are no such sums involving 3 or fewer summands. Indeed, the smallest such 3-summand sum is $136+247+58=441$. Likewise, for 2 summands or a single summand the smallest such sums are respectively $1357+2468=3825$ and 12345678.

**(b)** To prove all sums arising as described are divisible by 9, consider the following argument. The possible sums range in value between

$$36=(1+2+3+4+5+6+7+8)\cdot 10^0,$$

a sum of eight numbers, and

$$87654321=1\cdot 10^0+2\cdot 10^1+3\cdot 10^2+4\cdot 10^3+5\cdot 10^4+6\cdot 10^5+7\cdot 10^6+8\cdot 10^7,$$

a sum of a single number. In general, we will have a sum of $k$ summands, with $1\le k\le 8$. Denote these summands as $n_1$,$n_2$,...,$n_k$, and consider

$$n_1+n_2+\cdots+n_k=s_0\cdot 10^0+s_1 10^1+s_2 10^2+s_3 10^3+s_4 10^4+s_5 10^5+s_6 10^6+s_7 10^7,$$

where $s_p$ is the sum of the digits in the $n_k$ which are in the $(p+1)$st place value counting from right to left. As an example with $k=5$, consider

$$\underbrace{147+82+3+5+6}_{n_1+n_2+n_3+n_4+n_5}=\underbrace{(7+2+3+5+6)\cdot 10^0+(4+8)\cdot 10^1+1\cdot 10^2}_{s_0 10^0+s_1 10^1+s_2 10^2},$$

where for this example $s_3=s_4=s_5=s_6=s_7=0$. In general we must have $s_0+s_1+s_2+s_3+s_4+s_5+s_6+s_7=36$. Therefore,

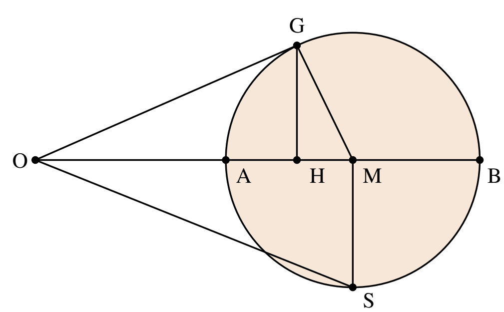
$$\begin{aligned}n_1+n_2+\cdots+n_k=36&+s_1(10^1-1)+s_2(10^2-1)+s_3(10^3-1)+s_4(10^4-1)\\&+s_5(10^5-1)+s_6(10^6-1)+s_7(10^7-1).\end{aligned}$$

All terms here are divisible by 9.

**2.** In the figure points $A$, $B$, $G$, $S$ are on a circle whose center is at $M$, and $O$ is a point on the extension of the diameter $AB$. Moreover, $OG$ is tangent to the circle, with both $GH$ and $SM$ perpendicular to the diameter $AB$. Suppose $a=\overline{OA}$ and $b=\overline{OB}$, with $0<a<b$. Express the inequalities

$$\overline{OH}<\overline{OG}<\overline{OM}<\overline{OS}$$

in terms of $a$ and $b$.



**Solution.** Start with the observation that

$$\overline{OM} = \frac{\overline{OA}+\overline{OB}}{2} = \frac{a+b}{2}.$$

Now let $r = \overline{MG} = \overline{MS} = \overline{MA} = \overline{MB}$ be the radius of the circle. Then

$$\overline{OB} - \overline{OA} = b - a = 2r,$$

showing that $r = \frac{1}{2}(b-a)$ and

$$\overline{OS} = \sqrt{\overline{OM}^2 + r^2} = \sqrt{\left(\frac{a+b}{2}\right)^2 + \left(\frac{b-a}{2}\right)^2} = \sqrt{\frac{a^2+b^2}{2}}.$$

Next, consider

$$\overline{OG} = \sqrt{\overline{OM}^2 - r^2} = \sqrt{\left(\frac{a+b}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2} = \sqrt{ab}.$$

Finally, notice that $\Delta HMG$ is similar to $\Delta GMO$. It follows that

$$\frac{\overline{HM}}{r} = \frac{r}{\overline{OM}}.$$

Since $\overline{HM} = \overline{OM} - \overline{OH}$, the last equation becomes

$$\frac{\overline{OM} - \overline{OH}}{r} = \frac{r}{\overline{OM}},$$

from which we find

$$\overline{OH} = \overline{OM} - \frac{r^2}{\overline{OM}} = \frac{a+b}{2} - \frac{2}{a+b}\left(\frac{b-a}{2}\right)^2 = \frac{2}{a+b}\left[\left(\frac{a+b}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2\right] = \frac{2ab}{a+b}.$$

Collecting the results, we see that, expressed in terms of $a$ and $b$, the stated chain of inequalities is the following.

$$\boxed{\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}}}$$

We have assumed that $0 < a < b$, but note that were $0 < a = b$, then all expressions would equal $a$. Finally note that we might express the leftmost expression in the chain as

$$\frac{2ab}{a+b} = \frac{1}{\dfrac{\frac{1}{a}+\frac{1}{b}}{2}},$$

which is the *harmonic mean* of $a$ and $b$. In order the expressions in the chain are the *harmonic mean*, *geometric mean*, *arithmetic mean*, and *quadratic mean* (or *root mean square*) of $a$ and $b$.

**3.** In the hexadecimal (base-16) number system a (nonstandard) notation for the base symbols is the following.

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, u, v, w, x, y, z$$

**(a)** Find the hexidecimal representation of the decimal number $(301.5)_{10}$.

**(b)** Find the decimal representation of the hexadecimal number $(z.\overline{u9v})_{16}$. You may give your answer as a fraction in simplest form.

**Solution.** For **(a)** the number $(301.5)_{10}$ has integer part $(301.0)_{10}$ and fractional part $(0.5)_{10}$. To convert the integer part to base-16, consider

$$\begin{aligned} 301/16 &= 18 \text{ remainder } 13 \\ 18/16 &= 1 \text{ remainder } 2 \\ 1/16 &= 0 \text{ remainder } 1. \end{aligned}$$

So the hexadecimal representation of $(301.0)_{10}$ is $(12x.0)_{16}$. As a double check, $1\cdot 16^2+2\cdot 16+13\cdot 16^0 = 256+32+13 \stackrel{\checkmark}{=} 301$. To convert the fractional part $(0.5)_{10}$ to hexadecimal, we simply note that

$$0.5\cdot 16 = 8+0,$$

showing $(0.5)_{10} = (0.8)_{16}$. We conclude that $(301.5)_{10} = (12x.8)_{16}$. For **(b)** notice that

$$16^3(z.\overline{u9v})_{16} = (zu9v.\overline{u9v})_{16}.$$

Therefore,

$$\begin{aligned} (16^3-1)(z.\overline{u9v})_{16} &= (zu9v.\overline{u9v})_{16} - (z.\overline{u9v})_{16} \\ &= (zu9v.0)_{16} - (z.0)_{16} \\ &= z\cdot 16^3 + u\cdot 16^2 + 9\cdot 16^1 + v\cdot 16^0 - z\cdot 16^0 \\ &= 15\cdot 4096 + 10\cdot 256 + 9\cdot 16 + 11 - 15 \\ &= 61440 + 2560 + 144 + 11 - 15 \\ &= 64155 - 15 \\ &= 64140, \end{aligned}$$

and we have

$$(z.\overline{u9v})_{16} = \tfrac{64140}{16^3-1} = \tfrac{64140}{4095} = \tfrac{4276}{273}.$$

**Remark.** Here is another approach to part **(b)** which involves geometric series.

$$\begin{aligned} (z.\overline{u9v})_{16} &= 15\cdot 16^0 + 10\cdot 16^{-1} + 9\cdot 16^{-2} + 11\cdot 16^{-3} + 10\cdot 16^{-4} + 9\cdot 16^{-5} + 11\cdot 16^{-6} + \cdots \\ &= 15 + \tfrac{10}{16} + \tfrac{9}{16^2} + \tfrac{11}{16^3} + \left(\tfrac{10}{16} + \tfrac{9}{16^2} + \tfrac{11}{16^3}\right)\tfrac{1}{16^3} + \left(\tfrac{10}{16} + \tfrac{9}{16^2} + \tfrac{11}{16^3}\right)\tfrac{1}{16^6} + \cdots \\ &= 15 + \left(\tfrac{10}{16} + \tfrac{9}{16^2} + \tfrac{11}{16^3}\right)\left(1 + \tfrac{1}{16^3} + \tfrac{1}{16^6} + \cdots\right). \end{aligned}$$

Now we compute $\frac{10}{16} + \frac{9}{16^2} + \frac{11}{16^3} = \frac{1}{4096}(2560+144+11) = \frac{2715}{4096}$, in order to write

$$\begin{aligned} (z.\overline{u9v})_{16} &= 15 + \tfrac{2715}{4096}\sum_{k=0}^{\infty}\left(\tfrac{1}{4096}\right)^k \\ &= 15 + \tfrac{2715}{4096}\tfrac{4096}{4095} \\ &= 15 + \tfrac{181}{273}. \end{aligned}$$

The middle equality follows from the formula for a convergent geometric series:

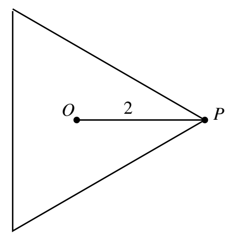
$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \qquad |r| < 1.$$

Therefore,

$$(z.\overline{u9v})_{16} = \tfrac{4095+181}{273} = \tfrac{4276}{273},$$

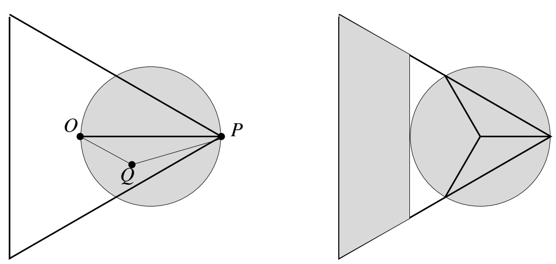
as found earlier.

**4.** Let $\mathcal{T}$ be a solid equilateral triangle, with center point $O$. Suppose the shown segment with endpoints $O$ and $P$ has length 2. Define the region $\mathcal{R} \subset \mathcal{T}$ to be all points $Q$ inside $\mathcal{T}$ such that the triangle $\Delta OPQ$ has an obtuse angle. What is the area of $\mathcal{R}$?




**Solution.** For points within the radius-1 circle shown in the left figure, $\angle OQP$ is obtuse, whereas if $Q$ were on the circle, then $\angle OQP = \frac{1}{2}\pi$. See the solution for **Problem 10, Round 1**! So part of the area of $\mathcal{R}$ stems from the intersection of this circle's interior with $\mathcal{T}$. This intersection (see the right figure) is comprised of a sector of area $\frac{1}{3}\pi$ and two isosceles triangles, each with area $\frac{1}{4}\sqrt{3}$. So the area of the intersection is $\frac{1}{3}\pi + \frac{1}{2}\sqrt{3}$.

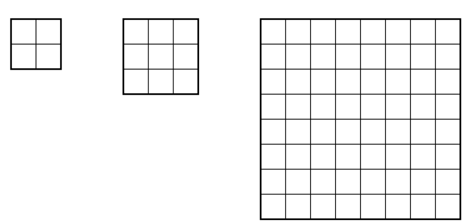
The remaining portion of $\mathcal{R}$ is the shaded 4-sided polygon also shown in the right figure. This polygon arises as the intersection of two equilateral triangles, a larger one (in fact $\mathcal{T}$ itself) with area $3\sqrt{3}$, and a smaller triangle with area $\frac{4}{3}\sqrt{3}$. Therefore, the area of the 4-sided polygon is $\frac{5}{3}\sqrt{3}$. The total area of $\mathcal{R}$ is then $\frac{1}{3}\pi + \frac{1}{2}\sqrt{3} + \frac{5}{3}\sqrt{3} = \frac{1}{3}\pi + \frac{13}{6}\sqrt{3}$.




**5.** Consider the game boards shown in the figure, respectively with 4, 9, and 64 squares or *cells.*

**(a)** For the 4-cell board how many possible ways can 2 cells be chosen from the board? If 2 cells of the 4-cell board are chosen at random, then what is the probability they will have a common side?

**(b)** Answer the same questions for the 9-cell board and the 64-cell board.



**Solution. (a)** (i) If we first choose the top-left cell, then there are 3 other ways the second cell can be chosen. (ii) If we first choose the top-right cell, there are 2 other new ways the second cell can be chosen; choice of the top-left cell as the second cell corresponds to a choice of 2 cells already considered in (i). (iii) If we first choose the bottom-right cell, there is 1 new way to choose the second cell. (iv) Finally, if we first choose the bottom left cell, then all possibilities have already been considered. So the number of possible ways to choose 2 cells from the 4-cell board is $3+2+1=6$. This is the binomial coefficient "4 choose 2":

$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = 6.$$

Neighoring cells share a common edge. Each cell of the 4-cell board has 2 neighbor cells, so the number of ways to choose 2 adjacent cells is (with the division by 2 to correct for over-counting)

$$\frac{4 \cdot 2}{2} = 4.$$

So the probability of choosing 2 adjacent cells from the 4-cell board is

$$\frac{4}{6} = \frac{2}{3}.$$

For **(b)** and the 9-cell board we can similarly reason brute-force. The number of ways to choose 2 cells is

$$8+7+6+5+4+3+2+1 = 36 \text{ or equivalently } \binom{9}{2} = \frac{9!}{2!(9-7)!} = 36.$$

On the 9-cell board the interior cell as 4 neighbor cells, 4 of the edge cells have 3 neighbor cells, and the 4 corner cells have 2 neighbor cells. So the number of ways to choose 2 adjacent cells is

$$\frac{1 \cdot 4 + 4 \cdot 3 + 4 \cdot 2}{2} = 12,$$

and the relevant probability is

$$\frac{12}{36} = \frac{1}{3}.$$

Finally, for the 64-cell board we have $8 \times 8 = 64$ total cells on the chess board. The total number of ways to choose 2 cells out of 64 cells is

$$\binom{64}{2} = \frac{64!}{2! \cdot 62!} = \frac{63 \cdot 64}{2} = 63 \cdot 32 = 2016.$$

Of the 64 total cells $6 \times 6 = 36$ cells (interior cells) have 4 neighbor-cells. Of the boundary cells, $6+6+6+6 = 24$ have 3 neighbor-cells. Only the four corner boundary cells have 2 neighbor-cells. The number of ways to choose 2 adjacent cells is then

$$\frac{36 \cdot 4 + 24 \cdot 3 + 4 \cdot 2}{2} = 112.$$

So the probability of choosing 2 adjacent cells from the 64-cell board is

$$\frac{112}{2016} = \frac{1}{18}.$$

**6.** Recall the absolute value function

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x \leq 0, \end{cases}$$

and consider an ordered pair $(x, y)$ of real numbers. Viewing $(x, y)$ as a two-component vector, its *weighted 1-norm* is

$$\|(x, y)\|_{(w_1, w_2)} = w_1|x| + w_2|y|,$$

where $w_1$ and $w_2$ are strictly positive real numbers call the *weights*.

**(a)** Sketch the region in the plane (in fact a polygon) whose points obey $\big\|(x,y)\big\|_{(4,2)} \le 10$.

**(b)** The region determined by $\big\|(|x|+1,|y|-2)\big\|_{(4,2)} \le 10$ is also a polygon. Specify it.

**Solution.** For **(a)** notice that the region determined by $\|(x,y)\|_{(4,2)} = 4|x| + 2|y| \le 10$ will be symmetric across both the $x$ and $y$-axis. So we need only determine those points in the first quadrant which obey the inequality. For non-negative $x$ and $y$ consider then $\|(x,y)\|_{4,2} = 10$ or

$$4x + 2y = 10.$$

This is line segment connecting the points $(0,5)$ and $(\frac{5}{2},0)$. The points in the first quadrant which lie below this line obey $4x+2y<10$. Reflection of this first-quadrant region across both axes yields the diamond region shown in Figure 1. The boundary segments of this diamond are determined by the following equations.

(i) $4x + 2y = 10$ (first quadrant for $0 \le x \le \frac{5}{2}$)
(ii) $-4x + 2y = 10$ (second quadrant for $-\frac{5}{2} \le x \le 0$)
(iii) $-4x - 2y = 10$ (third quadrant for $-\frac{5}{2} \le x \le 0$)
(iv) $4x - 2y = 10$ (fourth quadrant for $0 \le x \le \frac{5}{2}$)

For **(b)** first consider the simpler inequality $\|(x+1,y-2)\|_{(4,2)} = 4|x+1| + 2|y-2| \le 10$. This inequality determines the light-shaded diamond shown in Figure 2, namely the diamond in Figure 1 with its center translated from $(0,0)$ to $(-1,2)$. The translations of the corresponding boundary segments listed in (i) and (iv) above are the following.

(i′) $4(x+1) + 2(y-2) = 10$ or $4x + 2y = 10$ (for $-1 \le x \le \frac{3}{2}$)
(iv′) $4(x+1) - 2(y-2) = 10$ or $4x - 2y = 2$ (for $-1 \le x \le \frac{3}{2}$)

Segment (i′) connects the points $(0,5)$ and $(\frac{3}{2},2)$ in the first quadrant. Segment (iv′) connects the points $(\frac{1}{2},0)$ and $(\frac{3}{2},2)$ in the first quadrant. The light-shaded diamond centered at $(-1,2)$ in Figure 2 intersects the first quadrant in the dark-shaded region shown in the figure.

The final inequality to consider,

$$\|(|x|+1,|y|-2)\|_{4,2} = 4\big||x|+1\big| + 2\big||y|-2\big| \le 10,$$

involves $|x|$ and $|y|$; whence the region it specifies must be symmetric about both the $x$ and $y$ axes. So the region in question is obtained via reflection across the coordinate axes of the dark-shaded region in Figure 2. The resulting region is the 8-sided figure shown in Figure 3.

**7.** Let players $A$ and $B$ take turns flipping a fair coin with player $A$ going first. The players flip the coin until a tails occurs after a heads. The first player to toss tails immediately after a heads wins. Find the probability that player $A$ wins.

**Solution.** The nouns (and verbs) "toss" and "flip" are used as synonyms. Consider conditioning on the first two flips, and let $E$ be the event that player $A$ wins. We order the past-to-future toss outcomes left-to-right. Then, for example, $A$ tossing tails followed by $B$ tossing heads is represented as $TH$. Via the rules of conditional probability,

$$\begin{aligned} P(E) &= P(E|HH)P(HH) + P(E|HT)P(HT) + P(E|TH)P(TH) + P(E|TT)P(TT) \\ &= \tfrac{1}{4}\big[P(E|HH) + P(E|HT) + P(E|TH) + P(E|TT)\big], \end{aligned}$$

since for a fair coin $P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$. Note that $P(E|HT) = 0$, since in this case player $B$ wins. Next, note that $P(E|TT) = P(E)$, since the outcome $TT$ is equivalent to the game not having started. Therefore, the above formula reduces to

$$P(E) = \tfrac{1}{3}\big[P(E|HH) + P(E|TH)\big].$$

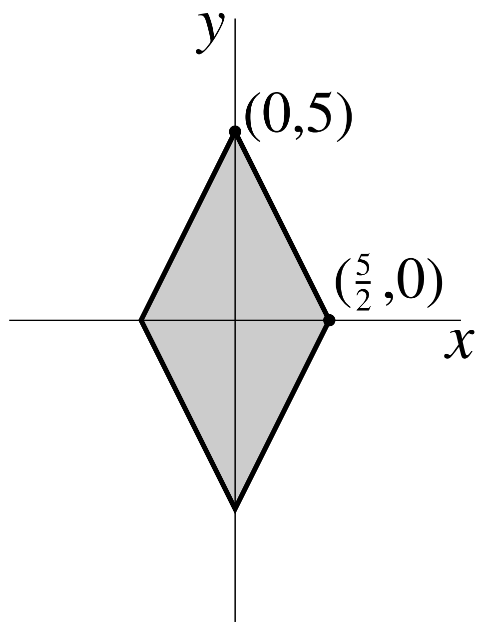



Figure 1. Shaded region for which $4|x| + 2|y| \leq 10$.

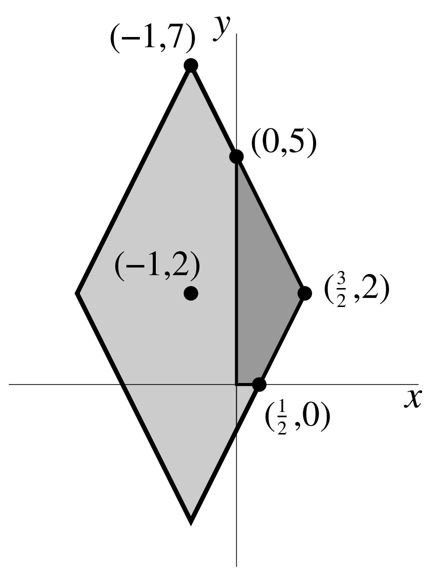



Figure 2. Shaded region for which $4|x + 1| + 2|y - 2| \leq 10$.

For the two conditional probabilities $P(E|HH)$ and $P(E|TH)$, we have a situation where player $B$ just tossed $H$, and player $A$ might win on their next toss. Set $p_1 = P(E|HH) = P(E|TH)$, so that from above

$$P(E) = \tfrac{2}{3}p_1.$$

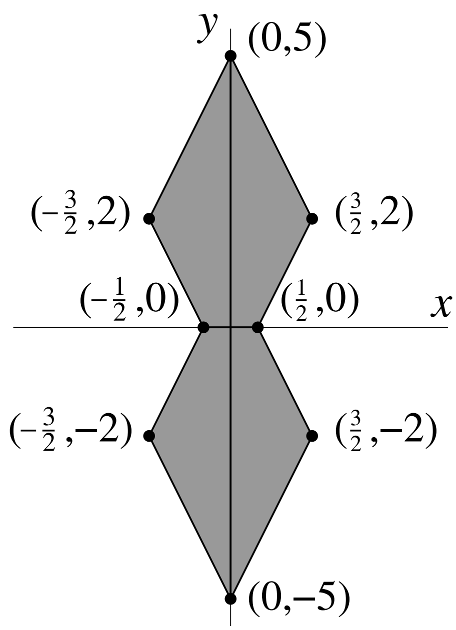



FIGURE 3. Shaded region for which $4||x|+1|+2||y|-2| \leq 10$.

We determine the probability $p_1$ by conditioning on the next toss. Note that if the next toss is an $H$, then player $B$ has the advantage and wins with probability $p_1$. In this case, player $A$ wins with probability $1-p_1$. Therefore,

$$p_1 = \tfrac{1}{2}(1-p_1) + \tfrac{1}{2},$$

showing $p_1 = \frac{2}{3}$ and $P(E) = \frac{4}{9}$.

**8.** Given $p(x) = \frac{1}{2}x^2 - x + \frac{1}{3}$, consider the quadratic equation $p(x) = 0$. Starting with $x_0 = 0$, the iterative scheme

$$x_{k+1} = x_k - \frac{\frac{1}{2}x_k^2 - x_k + \frac{1}{3}}{x_k - 1},$$

generates a *Newton sequence* $x_0, x_1, x_2, \ldots$ which converges to a root of the quadratic equation.

**(a)** Find the roots $x_-$ and $x_+$ of this quadratic equation, assuming $x_- < x_+$. Write down the first three terms $x_0, x_1, x_2$ of the Newton sequence.

**(b)** Assuming $0 \leq x_k < x_-$, show that (i) $x_k < x_{k+1}$ and (ii) $x_{k+1} < x_-$. *Hint: for (ii) consider* $p(x_k + (x_- - x_k))$.

**(c)** Specify the real numbers $a$ and $\theta$ in the following exact formula for the iterates in the sequence.

$$x_k = a\Big(\frac{1-\theta^{2^k-1}}{1-\theta^{2^k}}\Big)$$

**Solution.** **(a)** The roots of the quadratic equation $\frac{1}{2}x^2 - x + \frac{1}{3} = 0$ are

$$x_\pm = 1 \pm \sqrt{\tfrac{1}{3}},$$

and clearly $0 < x_- < x_+$. Starting with the "guess" (a more becoming term is "initial iterate") $x_0 = 0$, we find

$$x_1 = 0 - \frac{\frac{1}{3}}{-1} = \tfrac{1}{3}, \qquad x_2 = \tfrac{1}{3} - \frac{\frac{1}{2}\frac{1}{9} - \frac{1}{3} + \frac{1}{3}}{\frac{1}{3} - 1} = \tfrac{5}{12}.$$

**(b)** For any $x \in [0, x_-)$ we claim that

$$\text{(i) } x < x - \frac{\frac{1}{2}x^2 - x + \frac{1}{3}}{x - 1}, \qquad \text{(ii) } x - \frac{\frac{1}{2}x^2 - x + \frac{1}{3}}{x - 1} \in [0, x_-).$$

To show (i), let $x \in [0, x_-)$, and first write the polynomial $p(x) = \frac{1}{2}(x - x_-)(x - x_+)$ in factored form. This shows $p(x) > 0$ for $x < x_-$. Moreover,

$$0 \le x < x_- = 1 - \sqrt{\tfrac{1}{3}} \implies -1 \le x - 1 < -\sqrt{\tfrac{1}{3}}.$$

So we see for $x \in [0, x_-)$ that the right-hand side of the iteration has the form

$$x - \frac{\frac{1}{2}x^2 - x + \eta}{x - 1} = x - \frac{\text{strictly positive number}}{\text{strictly negative number}} > x,$$

establishing (i). To show (ii), follow the hint and write

$$\begin{aligned} 0 = p(x_-) &= p(x + (x_- - x)) \\ &= p(x + \Delta x) \\ &= \tfrac{1}{2}(x + \Delta x)^2 - (x + \Delta x) + \tfrac{1}{3} \\ &= p(x) + (x - 1)\Delta x + \tfrac{1}{2}\Delta x^2. \end{aligned}$$

Rearrangement then gives

$$\Delta x + \frac{p(x)}{x - 1} = -\frac{\frac{1}{2}\Delta x^2}{x - 1},$$

where the division by $x - 1$ is allowed since we showed above the $x - 1 < 0$ for $x \in [0, x_-)$. From the last formula,

$$(1) \qquad x_- - \Big(x - \frac{p(x)}{x - 1}\Big) = -\frac{\frac{1}{2}(x_- - x)^2}{x - 1} > 0,$$

again using $x - 1 < 0$. Therefore,

$$0 \le x - \frac{p(x)}{x - 1} < x_-,$$

where we have inserted the inequality on the left using result (i). This is result (ii).

**(c)** For $k = 1, 2, 3$ the stated formula becomes

$$(2) \qquad 0 = x_0 = a\Big(\frac{1 - \theta^0}{1 - \theta}\Big), \qquad \tfrac{1}{3} = x_1 = a\Big(\frac{1 - \theta}{1 - \theta^2}\Big), \qquad \tfrac{5}{12} = x_2 = a\Big(\frac{1 - \theta^3}{1 - \theta^4}\Big).$$

The first equation here is $0 = 0$, and so is content-less. We might speculate that $x_k$ approaches $x_-$ as $k \to \infty$, and consistently guess that $a = x_-$. With this assumption, the middle equation in (2) is

$$\tfrac{1}{3} = \Big(1 - \sqrt{\tfrac{1}{3}}\Big)\Big(\frac{1}{1 + \theta}\Big) \implies \theta = 2 - \sqrt{3}.$$

The same results can be found from the last two equations in (2). Indeed,

$$\tfrac{5}{4} = \frac{x_2}{x_1} = \frac{1 - \theta^3}{1 - \theta^4}\frac{1 - \theta^2}{1 - \theta} = \frac{\theta^2 + \theta + 1}{\theta^2 + 1} \implies \theta^2 - 4\theta + 1 = 0.$$

This quadratic equation is solved by $\theta_\pm = 2 \pm \sqrt{3}$. Now write the second equation of (2) as

$$a = \tfrac{1}{3}(1 + \theta).$$

Therefore, we find two solutions

$$\theta_- = 2 - \sqrt{3},\ a_- = 1 - \sqrt{\tfrac{1}{3}} \quad \text{and} \quad \theta_+ = 2 + \sqrt{3},\ a_+ = 1 + \sqrt{\tfrac{1}{3}}.$$

The first pair is the one arrived at earlier. Because $\theta_-^{-1} = \theta_+$, we have that

$$a_-\Big(\frac{1-\theta_-^{2^k-1}}{1-\theta_-^{2^k}}\Big) = a_-\theta_+\Big(\frac{\theta_+^{2^k-1}-1}{\theta_+^{2^k}-1}\Big) = a_+\Big(\frac{1-\theta_+^{2^k-1}}{1-\theta_+^{2^k}}\Big),$$

showing that the two solutions define the same Newton iterates.

**Remarks.** The results from **(b)** show that $x_0, x_1, x_2, \ldots$ is a strictly increasing sequence bounded above by $x_-$. It follows (by the *Monotone Convergence Theorem*) that the sequence converges to a number $r \le x_-$, where $r$ is the least upper bound of the sequence. However, only $r = x_-$ is consistent with the form of the iteration; indeed $-p(x_k)/(x_k - 1) = x_{k+1} - x_k \to 0$ must hold as $k \to \infty$. Since $p(x)$ is a continuous function, this implies $p(r) = 0$, and so $r = x_-$. Alternatively, one may use (1) to write

$$x_- - x_{k+1} = \frac{\frac{1}{2}(x_- - x_k)^2}{1 - x_k} < \frac{\frac{1}{2}(x_- - x_k)^2}{1 - x_-}.$$

This shows that the errors are positive, and each is proportional to the square of the last. Note also $x_- - x_0 = x_- < 1$. We conclude $x_- - x_k \to 0^+$ as $k \to \infty$.

For **(c)** let us the derive the formula with $(\theta_-, a_-)$ for the iterates from scratch. Notice that

$$p(r) = \tfrac{1}{2}(x - x_-)(x - x_+), \qquad x - 1 = \tfrac{1}{2}\big[(x - x_+) + (x - x_-)\big].$$

So the iteration is

$$x_{k+1} = x_k - \frac{(x_k - x_-)(x_k - x_+)}{(x_k - x_+) + (x_k - x_-)}.$$

From this equation we get both

$$x_{k+1} - x_- = \frac{(x_k - x_-)^2}{(x_k - x_+) + (x_k - x_-)}, \qquad x_{k+1} - x_+ = \frac{(x_k - x_+)^2}{(x_k - x_+) + (x_k - x_-)},$$

and together these yield

$$\frac{x_{k+1} - x_-}{x_{k+1} - x_+} = \frac{(x_k - x_-)^2}{(x_k - x_+)^2}.$$

From the last formula and induction on $k$

$$\frac{x_k - x_-}{x_k - x_+} = \theta^{2^k}, \qquad \theta = \frac{x_-}{x_+} = \frac{\sqrt{3}-1}{\sqrt{3}+1} < 1.$$

Rearrangement gives the formula

$$x_k = x_-\Big(\frac{1-\theta^{2^k-1}}{1-\theta^{2^k}}\Big).$$

So $a = 1 - \sqrt{\frac{1}{3}}$ and $\theta = (\sqrt{3}-1)/(\sqrt{3}+1) = 2 - \sqrt{3}$, corresponding to the $(\theta_-, a_-)$ representation found earlier.