UNM – PNM STATEWIDE MATHEMATICS CONTEST LII

February 1, 2020   Second Round   Three Hours

1. Calculate the sum of the roots for $|2x - 4| = 5$.

   Answer: 4

   Solution: We have two roots: $2x_1 - 4 = 5$ and $2x_1 - 4 = -5$. So we have $2(x_1 + x_2) = 8$ which gives $x_1 + x_2 = 4$.

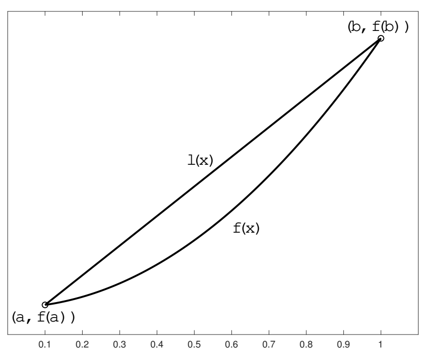
2. Show that there exist two numbers, among any $k + 1$ numbers, such that their difference is divisible by k.

   Solution: We have k possible remainders when dividing by k. Since there are $k+1$ numbers, two of them must have the same remainder when they are divided by k (this is also called the pigeon-hole principle). Let these numbers be x and y. Then,

   $$x = kn_1 + r \qquad \text{and} \qquad y = kn_2 + r$$

   for some integers n_1 and n_2. Hence $x - y = kn_3$ for some integer n_3.

3. For any $p > 1$ and $a, b > 0$, we have the property that the graph of the function $f(x) = x^p$ for $a \leq x \leq b$ lies below the line segment joining $(a, f(a)\ )$ and $(b, f(b))$. That is, let $(x, l(x))$ denote the points on the line segment joining $(a, f(a))$ and $(b, f(b))$. Then we have $f(x) < l(x)$ for $a < x < b$, as illustrated in the figure below for $a = 0.1, b = 1$ and $p = 2$.



   Use this fact to show that

   $$(2020)^{2020} < 2^{2019}(2000^{2020} + 20^{2020})$$

   Solution: Let $f(x) = x^{2020}$, and consider the property mentioned in the problem at the mid-point of the line segment for $a = 2000, b = 20$,

   $$f(2020/2) = f((2000 + 20)/2) < \frac{f(2000) + f(20)}{2}$$

   Using the definition of f

   $$(2020/2)^{2020} < (1/2)(2000^{2020} + 20^{2020})$$

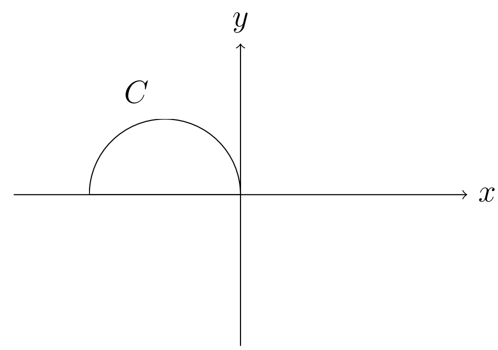
   from which the result follows immediately.

   Note: The property that the graph of the function $f(x) < l(x)$ for $a \leq x \leq b$ is known as convexity of the function $f(x)$ for $a \leq x \leq b$.

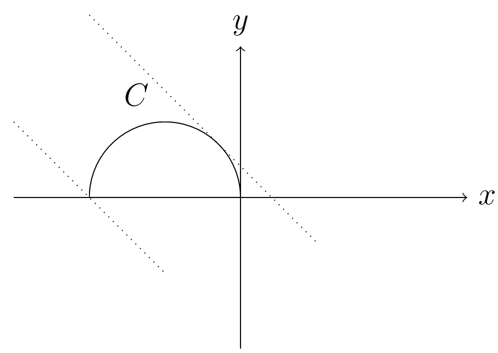
4. Let us consider a curve C and a line l in $\mathbb{R}^2$. The equation of the curve C is $y = \sqrt{-x^2 - 2x}$ and the equation of the line l is $x + y - m = 0$. For a particular value of m, l and C may either not intersect, or interact at one point, or intersect at two different points. Determine all the possible m such that C and l have two different intersection points.

   Answer: $0 \leq m < \sqrt{2} - 1$

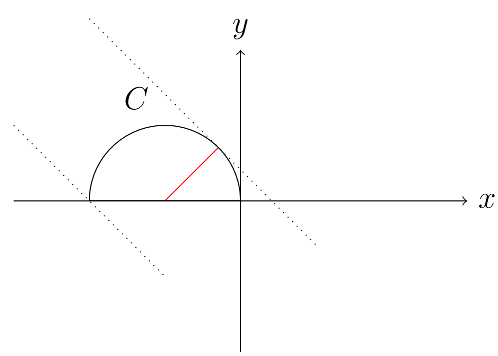
   Solution: We can rewrite the equation of C as $x^2 + 2x + y^2 = 0$ in the upper half plane. This is in fact a part of a circle in the upper half plane because we can see that the previous equation is equivalent to $(x + 1)^2 + y^2 = 1$.



   Now by changing the value of m we will get a family of lines. It is easy to see that these lines will either intersects C at two points, 1 point, or no point. These are two cases where these lines start to intersect l at 1 point: 1. $x + y - m = 0$ passes through $(-2, 0)$; 2. $x + y - m = 0$ passes through $(-1 + 1/\sqrt{2}, 1/\sqrt{2})$.



Let us explain the second line equation in more details. In order to determine the equation of the second line, we only need to know the intersection point of the second line and C. To calculate the coordinates of the point, we draw a segment between the center of the circle and the point.



Notice that this line segment must be perpendicular to the second line. Suppose the coordinates of the intersection point of the second line and the line segment is (x_0, y_0). We have

$$(x_0+1)^2 + y_0^2 = 1$$
$$y_0 = x_0 + 1$$

And we obtain $(x_0, y_0) = (-1 + 1/\sqrt{2}, 1/\sqrt{2})$. We see now that we only want to consider the value of m in $(-2, \sqrt{2}-1)$. Notice that in order to have two intersection points, $x + y - m = 0$ must be in the right of $x + y = 0$. Thus the range of m is $[0, \sqrt{2}-1)$.

5. Let

$$x, a_1, a_2, a_3, y \qquad \text{and} \qquad b_1, x, b_2, 2y, b_3$$

be the terms from two geometric sequences, where $x \neq 0$ and $y \neq 0$. Calculate $\dfrac{b_3 a_1^8}{a_2^8 b_1}$.

Answer: 4

Solution: We have $b_3/b_1 = (2y/x)^2$ and $a_2/a_1 = (y/x)^{1/4}$. Hence

$$\frac{b_3 a_1^8}{a_2^8 b_1} = \frac{b_3/b_1}{(a_2/a_1)^8} = 4\frac{(y/x)^2}{(y/x)^2} = 4$$

6. Let $x = 0.82^{0.5}, y = \sin(1), z = \log_3(\sqrt{7})$. Determine the largest and smallest numbers among x, y, z.

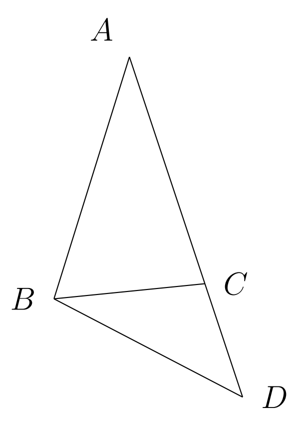
Answer: Smallest is y. Largest is x.

Solution: $x = 0.82^{0.5} > 0.81^{0.5} = 0.9$. $y = \sin 1 < \sin(\pi/3) = \frac{\sqrt{3}}{2} < \frac{1.74}{2} = 0.87$. For z, we notice that $3^7 < 7^4$ and $3^9 < 7^5$, so z in between $\frac{7}{8} = 0.875$ and 0.9. Thus x is the largest number and y is the smallest number.

7. Let $\triangle ABC$ be an acute triangle. $\angle C = 2\angle A$ and $2|AC| = |AB| + |BC|$. Calculate $\sin \angle A$.

Answer: $\frac{\sqrt{7}}{4}$

Solution: Let us extend AC to D such that $|BC| = |CD|$. Now draw a straight line between B and D. Notice that $\triangle ABD$ and $\triangle BCD$ are isosceles triangles.



Let $|AB| = c, |BC| = a, |AC| = b$, then $|CD| = a$ and $|BD| = c$. In $\triangle ADB$, $\angle A = \angle D$ and by the law of cosines,

$$\cos \angle D = \frac{(b+a)^2 + c^2 - c^2}{2(b+a)c} = \frac{b+a}{2c}$$

Similarly, in triangle $\triangle BCD$,

$$\cos\angle D = \frac{c}{2a}.$$

Thus we have $\frac{a+b}{c} = \frac{c}{a}$ which gives $a^2+ab = c^2$. Since $2b = a+c$, we obtain $2a^2+a(a+c) = 2c^2$. Let $k = \frac{c}{a}$. We thus have $2k^2 - k - 3 = 0$. That is $(2k-3)(k+1) = 0$. We conclude $k = \frac{3}{2}$. Thus $\cos\angle A = \frac{3}{4}$ and $\sin\angle A = \frac{\sqrt{7}}{4}$.

8. Let x_1, x_2, x_3 be the three roots of $x^3 - x + 1 = 0$. Calculate $x_1^5 + x_2^5 + x_3^5$.

   Answer: -5

   Solution: By assumption we have $0 = x^3 - x + 1 = (x-x_1)(x-x_2)(x-x_3)$. This implies

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1x_2 + x_1x_3 + x_2x_3 &= -1 \\ x_1x_2x_3 &= -1 \end{aligned}$$

Then

$$\begin{aligned} &x_1^5 + x_2^5 + x_3^5 \\ =&x_1^5 + x_2^5 - (x_1+x_2)^5 \\ =&x_1^5 + x_2^5 - x_1^5 - 5x_1^4x_2 - 10x_1^3x_2^2 - 10x_1^2x_2^3 - 5x_1x_2^4 - x_2^5 \\ =&-5x_1^3x_2(x_1+x_2) - 5x_1^2x_2^3(x_1+x_2) - 5x_1x_2^3(x_1+x_2) \\ =&5x_1^3x_2x_3 + 5x_1^2x_2^3x_3 + 5x_1x_2^3x_3 \\ =&5x_1x_2x_3(x_1^2 + x_1x_2 + x_2^2) \\ =&-5(x_1^2 + x_1x_2 + x_2^2) \\ =&-5((x_1+x_2)^2 - x_1x_2)) \\ =&-5(-x_3(x_1+x_2) - x_1x_2) \\ =&-5 \end{aligned}$$

9. What is the remainder when the number

$12+11(13^1)+10(13^2)+9(13^3)+8(13^4)+7(13^5)+6(13^6)+5(13^7)+4(13^8)+3(13^9)+2(13^{10})+13^{11}$

   is divided by 6?

   Answer: 0

   Solution:

   We first show that any positive integer power of 13 has remainder 1 when divided by 6, i.e. $13^n = 6\times k+1$ for some integer k. We have $13 \bmod 6 = 1$. Now consider 13^n for any positive integer n. Then,

$$\begin{aligned} 13^n &= (13^{n-1})(13) \\ &= (6\times k_1 + 1)(6\times 2+1) \qquad \text{(by the induction hypothesis)} \\ &= (6\times k_2 + 1) \qquad \text{(for some integer } k_2) \end{aligned}$$

   Now, let x be the number given in the problem. Consider $y = 1+2+3+4+5+6+7+8+9+10+11+12 = 78$. Then,

$$x - y = 11(13^1-1) + 10(13^2-1) + \cdots + (13^{11}-1)$$

Note that each term in $x-y$ is of the form $a(13^n-1) = a(6k+1-1) = a(6k)$ for some integers a, k and hence is divisible by 6. Thus, $x-y$ is divisible by 6. $y = 78$ is also divisible by 6, and so $x = (x-y)+y$ is also divisible by 6. Hence the remainder is 0.

10. A five-digit number is formed by randomly choosing (with no repetitions) a number from $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ for each digit. For example, 10234 is a valid number, while 12325 is not (since 2 is repeated in the latter number).
   (a) Show that the number thus formed is divisible by 9 if and only if the sum of its digits is divisible by 9 (e.g. $92034 \bmod 9 = 0$ and $(9+2+0+3+4) \bmod 9 = 18 \bmod 9 = 0$).
   (b) What is the probability that the five-digit number thus formed is divisible by 90?

   Answer: (b) 1/90

   Solution: Let the number be $x = 10,000a + 1,000b + 100c + 10d + e$.

   (a) We have,

$$x = 10,000a + 1,000b + 100c + 10d + e = (9999a + 999b + 99c + 9d) + (a + b + c + d + e)$$

   The first term in the parenthesis on the right-hand-side is divisible by 9, and so x is divisible by 9 if and only if $a+b+c+d+e$ is.

   (b) The total number of possible numbers that can be formed are $10 \times 9 \times 8 \times 7 \times 6 = 30,240$. Now, to be divisible by $90 = 9 \times 2 \times 5$, we must have $e = 0$. Thus, the four digit number $1,000a + 100b + 10c + d$ must be divisible by 9 and neither of $a, b, c, d = 0$. Now,

$$1,000a + 100b + 10c + d = (999a + 99b + 9c) + (a + b + c + d)$$

   From part (a) we have $a+b+c+d$ is divisible by 9. We also have,

$$a + b + c + d \le 9 + 8 + 7 + 6 = 30$$
$$a + b + c + d \ge 1 + 2 + 3 + 4 = 10$$

   Thus, $a+b+c+d$ is either 18 or 27.
   Lets consider 18 first. The following tuples containing 9 sum to 27 $(9, 6, 2, 1), (9, 5, 3, 1), (9, 4, 3, 2)$ giving a total of $3 \times 4!$ possibilities. Proceeding in a similar fashion, following tuples contain 8, but no 9: $(8, 7, 2, 1), (8, 6, 3, 1), (8, 5, 3, 2), (8, 5, 4, 1)$. The following tuples contain 7, but no 8 or 9: $(7, 6, 3, 2), (7, 6, 4, 1), (7, 5, 4, 2)$. For the remaining numbers, there is only one possibility: $(6, 5, 4, 3)$. Thus, for $a+b+c+d = 18$ we have a total of $11 \times 4!$ possibilities. For $a+b+c+d = 27$ we have the tuples $(9, 8, 6, 4), (9, 8, 7, 3), (9, 7, 6, 5)$ for a total of $3 \times 4!$ possibilities. Thus, the total number of numbers which are divisible by 90 are $14 \times 4! = 336$. Thus the probability is $336/30240 = 1/90$.