UNM - PNM STATEWIDE MATHEMATICS CONTEST XLVIII

November 6-9, 2015 First Round Three Hours

1. Among 50 students in the world languages club, 20 know Spanish, 15 know French and 10 know German. If 5 know both Spanish and French and 2 know French and German but none know both Spanish and German, how many students don't know any of these languages?

   Answer: 12.

   Solution: Let A be the set of students knowing Spanish, B, the set of students knowing French and C be the set of students knowing German. The number of students knowing at least one of Spanish, French or German is $|A \cup B \cup C|$. Using the principle of inclusion and exclusion, $|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C| = 20 + 15 + 10 - 5 - 2 = 38$. But we are searching for the number of students that don't know any of these languages and this number will be $50 - 38 = 12$.

2. A team of 5 students are chosen to work on a research project lead by a chemical engineer. If 4 chemistry students, 3 physics students and 2 math students express interest in working with the professor, how many ways can the professor choose 5 of the students if he wants at least 2 chemistry students on the team?

   Answer: 105 (60 for exactly two chemistry students).

   Solution: We will denote by $C(n,k) = \frac{n!}{(n-k)!k!}$, where $k! = 1 \cdot 2 \cdot 3 \dots k$, the number of ways one can choose k objects (in our case people) out of given n different objects.

   Consider first the case when there are exactly two chemistry students. Since there will be 5 students on the team and 2 are chemistry students, the professor has $C(4,2) = 6$ ways (the number of ways we can choose two people out of four) of choosing the chemistry students and $C(5,3) = 10$ ways of choosing the remaining three students on the team. So there are $6 \cdot 10 = 60$ ways the professor could choose his team with exactly two chemistry students.

   Thus, the professor can choose a team of five people with at least two chemistry students in $C(4,2)C(5,3) + C(4,3)C(5,2) + C(4,4)C(5,1) = 60 + 40 + 5 = 105$ ways.

3. A student is offered two different after school jobs. One pays \$8 an hour, the other pays \$1 for the first hour, but the hourly rate increases by \$1 dollar for each additional hour worked. At least how many hours must the student work, so that it will make it worth his while to take the second job?

   Answer: Both or either of 15 or 16: 16 to make more (15 to break even).

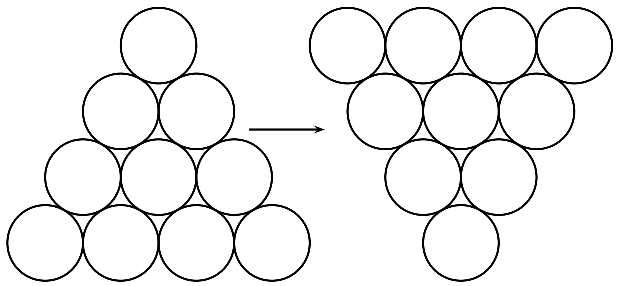
   Solution: If the student wants to make more money with the second job, he will need to work n hours such that
$$1 + 2 + 3 + \cdots + n > 8n.$$
   Since $1 + 2 + \cdots n = \dfrac{n(n+1)}{2}$, then $n(n+1) \geq 16n$ or $n^2 - 15n > 0$. To break even he has to work 15 hours, but to make more the student must work at least 16 hours to make it worth his while to take the second job.

4. Let $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$ and $g(x) = x^3 + x^2 - 1$, find polynomials $q(x)$ and $r(x)$ with integer coefficients such that $f(x) = g(x)q(x) + r(x)$ and the degree of $r(x)$ is smaller than 3.

   Answer: $q(x) = x^2 + 1$ and $r(x) = x^2 + x + 2$.

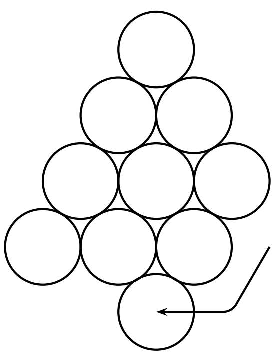
   Solution: Using long division we see that $x^5 + x^4 + x^3 + x^2 + x + 1 = (x^2 + 1)(x^3 + x^2 - 1) + x^2 + x + 2$ so $q(x) = x^2 + 1$ and $r(x) = x^2 + x + 2$. If you don't remember the long division algorithm, you could write
$$x^5 + x^4 + x^3 + x^2 + x + 1 = (a_1x^2 + a_2x + a_3)(x^3 + x^2 - 1) + b_1x^2 + b_2x + b_3$$
   and then solve for the unknown coefficients.

5. Ten congruent discs are packed forming an "equilateral triangle" in the xy coordinate plane whose base is parallel to the x-axis and the third vertex is above the base. You are asked to slide some of the discs to form a new "triangle" from the old one such that the new "triangle" has its base again parallel to the x-axis but now the remaining vertex is below the new base. A picture is provided below to assist you with the orientation of the initial and final "triangles". What is the minimal number of slides a student must perform to produce this new "triangle"?

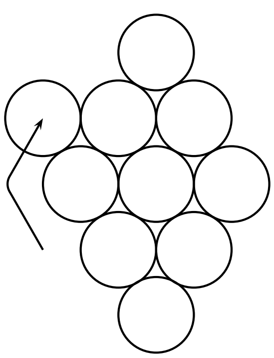


Answer: 3.

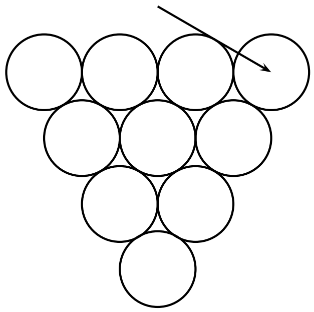
Solution: With one slide, we come to the following configuration.



After the second slide:

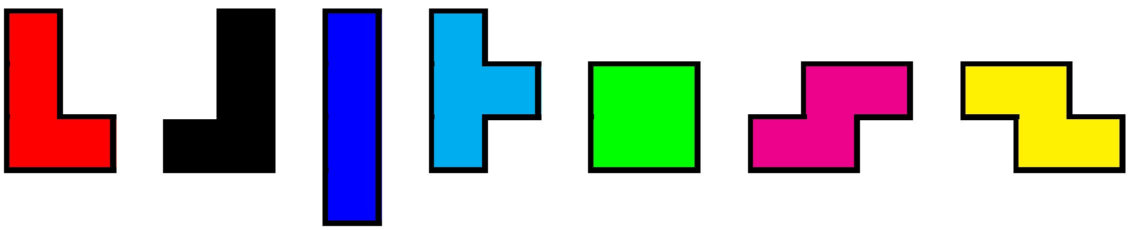


After the third slide:



Notice 2 or fewer moves does not produce a triangle.

6. In the game of tetris, there are 7 distinct polyomino pieces each constructed from four unit squares.



Each piece can be rotated 90, 180, or 270 degrees. List all values of n for which we can construct a rectangle of area $4n$ using *distinct* tetris pieces so that no pieces overlap and the rectangle has no holes?

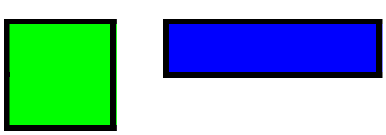
Answer: $\{1, 3, 4, 5, 6\}$.

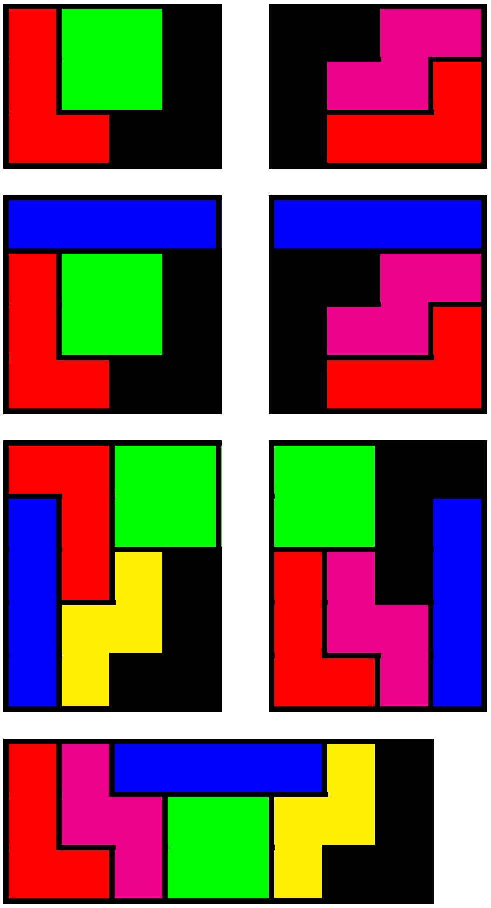
Solution: Since there are 7 distinct tetris pieces, the n can be at most 7. There can be no rectangle for $n = 2$ and $n = 7$ while for the remaining possible n's it is enough to give an examples which we do further below. Let us show first that there is no rectangle for $n = 7$. In fact,

We start by showing that any of the sought rectangles cannot use the "T"-shaped tetris piece. Indeed, let R be a rectangle constructed using different tetris pieces and suppose that the "T"=shaped piece is one of the used pieces. Let us color the unit squares of R black and white so that R looks like a (non-square) chess board. In other words, let us color the unit squares of R so that side-adjacent unit squares are of different color. Now, notice that each of the given tetris pieces will end-up having the same number of black and white unit squares, except the "T" shaped piece which will have three squares of one color and one square of the other color. By switching black and white color we can always reduce to the case where the "T" piece has three black and one white colored squares. But this leads to a contradiction since then R will have more black unit squares than white unit squares, which is a contradiction since R is constructed of even number unit squares (the area is an even number) so R has even number of rows (or columns) and two adjacent rows have the same number of black and white unit squares.

The above argument shows, in particular, that there is no rectangle for $n = 7$. By a direct quick check we see that there is no solution also for $n = 2$.

Below we exhibit rectangles of areas 4, 12, 16, 20 and 24, i.e., for $n = 1, 3, 4, 5, 6$.





7. The bottom of a swimming pool needs to be retiled. The surface that needs new tiles is a rectangle with sides of lengths 40.04m by 25.20m. In order to finish the job faster the contractor wants to use as large tiles as possible. They can order any size square tile of integer length in centimeters. What is the size of the tile the contractor ordered. Note: One meter has 100 centimeters, 1m=100cm.

   Answer: 28.

   Solution: $40.04m = 4004cm$ while $25.20m = 2520cm$. Since $4004 = 2^2 \cdot 7 \cdot 11 \cdot 13$ and $2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$ the largest square of integer dimension (in centimeters) that tiles the bottom of the pool has length $2^2 \cdot 7 = 28$ centimeters. Indeed, if a square with side of length k tiles the pool than if we use m tiles along the length and n tiles along the width of the pool we must have $km = 4004$ and $kn = 2520$. So the largest k we can take is the greatest common divisor of the numbers 4004 and 2520.

8. A triangle has sides all of whose lengths are integers. If two of its sides are 2 and 4, what are the possible lengths of its other side?

   Answer: $3, 4$ or 5.

   Solution: The sides are of length 3, 4 or 5. By the triangle inequality, if a, b and c are the lengths of the sides of the triangle, then $a + b > c$, $a + c > b$ and $b + c > a$. Let $a = 2$ and $b = 4$, then $a + b = 6 > c$. So $c \in \{1, 2, 3, 4, 5\}$. But we also must have $a + c > 4$ so $c > 2$. The lengths of the side will thus be $3, 4$ or 5.

9. What must the radius of a sphere be so that its volume is equal to the volume of a triangular prism of height two and whose base is an equilateral triangle whose sides are all 1?

   Answer: $r = \dfrac{\sqrt{3}}{2\sqrt[3]{\pi}}$.

Solution: $r = \dfrac{\sqrt{3}}{2\sqrt[3]{\pi}}$. The volume of a sphere is $V_S = 4/3\pi r^3$. The volume of a prism is $V_P = Bh$ where B is the area of the base. The area of the base is $B = \dfrac{1}{2} \cdot 1 \cdot \dfrac{\sqrt{3}}{2}$. So $V_P = \dfrac{\sqrt{3}}{4} \cdot 2 = \dfrac{\sqrt{3}}{2}$. Setting $V_S = V_P$, we see that $r^3 = \dfrac{3\sqrt{3}}{8\pi}$. Thus $r = \dfrac{\sqrt{3}}{2\sqrt[3]{\pi}}$.

10. Four ants A, B, C and D are out in a flat field when they realize that they cannot remember where their den is located. Each one of them decides to rely on the memory of the ant it sees directly ahead, so accordingly it decides to always walk directly towards this ant. It just so happens that at a certain moment the ants are at the corners of a square with sides of length 2 meters so that $AB = BC = CD = DA = 2$m (AB denotes the distance between ants A and B etc.), with ant A looking directly at ant B, who is looking at ant C, who is looking at ant D, who is looking at ant A. Suppose each of the ants moves at a constant speed of 3 cm/sec with ant A always moving directly towards ant B, who is moving towards C, who is moving towards ant D, who is moving towards ant A. How long will it take before the four ants meet if they meet at all?

Answer: $200/3 = 66.6(6) = 66\frac{2}{3}$.

Solution: From the symmetry, the ants will always be at the four vertices of a square with the same center, but the square is shrinking as well as rotating due to the motion of each of the ants. Since ant A is always moving perpendicularly to the line formed by ants B and C while B is moving toward ant C the distance between ants A and B is shrinking at the speed at which A is moving towards B, which is 2cm/s. Same applies to the "other" sides of the square formed by the ants. After 200/3s the square has shrunk to a point with all the ants having spiralled into the center.

We also can solve the problem analytically. One possible solution using complex numbers is as follows. Let $z_j(t)$, $j = 0, 1, 2, 3$, denote the position of the j-th ant at time t seconds after they start moving towards each other assuming z_0, z_1, $z - 2$ and z_3 are at the vertices of a square going counter-clock wise. Here $z_j = x_j + iy_j$ is a complex number representing the point (x_j, y_j) in the Euclidean plane. We can assume that the center of the coordinate system is at the center of the square. Let $z_0 = re^{i\phi}$ where r and ϕ are functions of time. Since the ants are at the vertices of a square at all times we have at any time $z_j = z_0 e^{i(j\pi/2)}$, $j = 1, 2, 3$. Now use that the velocity of ant z_0 is perpendicular to the direction determined by z_1 and z_2, hence z_0' is along the direction of z_o rotated $3\pi/4$ counter-clock wise. Thus we have

$$z_1 = z_0 e^{i(\pi/2)} \qquad z_0' = -kz_o e^{-i\pi/4}.$$

Also, as the ants are moving at a constant speed $a = 3$cm/s we have

$$a = |z_0'| = kr, \quad \text{i.e.} \qquad k = a/r = a/|z_0|.$$

Since $z_1 - z_0 = (1 - i)z_0$ it follows

$$\frac{d}{dt}|z_1 - z_0| = \sqrt{2}\, r'.$$

We can find r' on one hand by differentiating $z_0 = re^{i\phi}$, while on the other use the formula for z_0' given above, which yields

$$z_0' = r'e^{i\phi} - i\phi' re^{i\phi} = -kz_o e^{-i\pi/4} = -a/r\, z_o e^{-i\pi/4} = -ae^{i\phi}(1 - i)/\sqrt{2}.$$

Dividing by $e^{i\phi}$ we come to

$$r' - i\phi' r = -a(1 - i)/\sqrt{2},$$

which shows $r' = -a/\sqrt{2}$, hence $\frac{d}{dt}|z_1 - z_0| = -a$. In other words, we showed that distance between ant z_0 and $_1$ is shrinking at the speed at which z_0 is moving towards z_1, which is 3cm/s.