SOLUTIONS

UNM - PNM STATEWIDE MATHEMATICS CONTEST XLVII

February 7, 2015 Second Round Three Hours

1. In the sequence

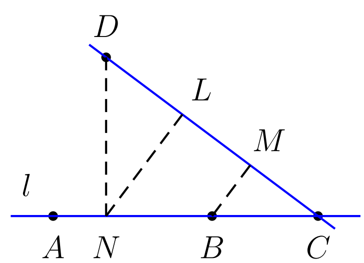
$$1, 2, 2, 4, 4, 4, 4, 8, 8, 8, 8, 8, 8, 8, 8, \cdots$$

what number occupies position 2015?

Solution: This sequence of numbers corresponds to $a_n = 2^k$, where k is the largest power of 2 appearing in the expansion n by powers of 2. Note that $a_3 = 2$ and $3 = 2^1 + 2^0 = 2 + 1$. Similarly $a_1 0 = 8$ and $10 = 2^3 + 2^1 = 8 + 2$. We need only look at the expansion of 2015 by 2: $2015 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0$ to see that the 2015th number is $2^{10} = 1024$.

2. Show that if S is a set of finitely many non-collinear points in the plane (i.e., not all of the points are on the same line), then there is a line which contains exactly two of the points of S. Is the claim true if S has infinitely many points? Hint: Use an extremal configuration.

Solution: This is a problem for which it is helpful to consider an extremal configuration. Suppose, the claim is not true, i.e., every line which contains two of the points of S contains a third one. We know that the points of S are noncollinear, hence taking into account that S has finitely many points there is a finite set of positive distances between points of S and lines containing at least two (hence three) points of S. A useful extremal configuration here is to consider a point $D \in S$ and a line l containing at least three points of S realizing the minimum of these distances. In other words given any line l' on which we can find at least three points of S and any point $D' \in S$ but $D' \in l'$ we have that the distance from D to l is not larger than the distance of D' to l'. Let A, B and C be three points of S lying on l. After possibly renaming them we can suppose that B and C are on the same side of the perpendicular from D to l, see picture below and note that B could be N.



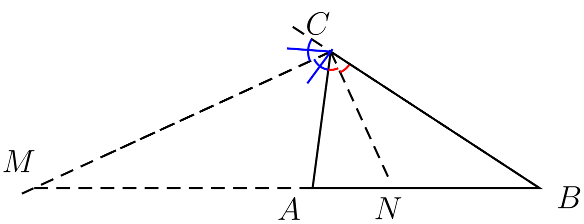
Notice that $BM \leq NL < DN$ which is a contradiction with the minimality of the distance between D and l. Therefore, there is a line which contains exactly two points of S.

The claim is not true for a set of infinitely many points. A counterexample is given, for example, by the two dimensional plane.

3. Show that the bisect of an angle in a triangle divides the opposite side in segments whose lengths have the same ratio as the ratio of the adjacent sides,

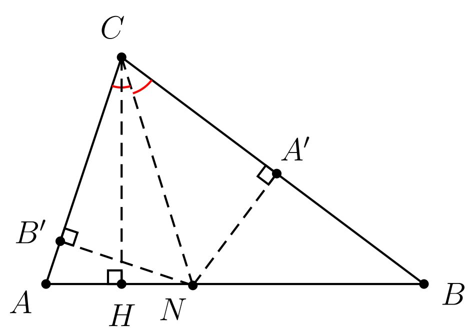
$$AN/NB = CA/CB$$

in the picture below. NOTE: The same is true for the bisector of an exterior angle of a triangle, i.e., it divides the opposite side externally into segments that are proportional to the adjacent sides. You do not have to write a proof of this fact.

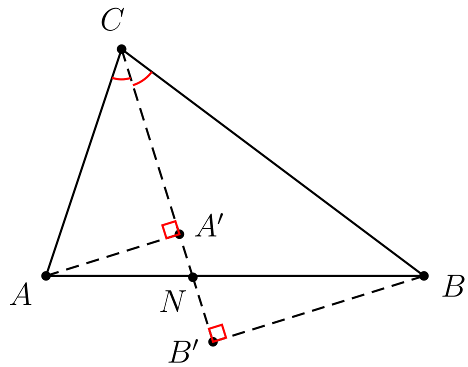


Solution: One way to prove the claim is to notice that any point on the bisect of an angle is at equal distances to the sides of the angle, see picture. We can write the areas of $\triangle ACN$ and $\triangle BCN$ in two ways, $A_{\triangle ACN} = \frac{CH \cdot AN}{2} = \frac{NB' \cdot AC}{2}$ and $A_{\triangle BCN} = \frac{CH \cdot BN}{2} = \frac{NB' \cdot BC}{2}$. Therefore,

$$\frac{A_{\triangle ACN}}{A_{\triangle BCN}} = \frac{AN}{BN} = \frac{AC}{BC}.$$



Another solution relies on the similar triangles (see picture below) $\triangle AA'N \approx \triangle BB'N$ and $\triangle AA'C \approx \triangle BB'C$ to see $\frac{AA'}{BB'} = \frac{AN}{BN}$ and $\frac{AA'}{BB'} = \frac{AC}{BC}$. The last two equalities imply $\frac{AN}{BN} = \frac{AC}{BC}$ which is what we needed to prove.



4. There are 12 coins in a parking meter and we know that one of them is counterfeit. The counterfeit coin is either heavier or lighter than the others. How can we find the fake coin and also if it is heavier or lighter in three weighings using a balance scale? Hint:4=3+1.

   Solution: This is a well known problem. You can find a solution to the stated and a more general problem here

   http://www.cut-the-knot.com/blue/OddCoinProblemsShort.shtml. Below we give the solution of the 12 coin problem. Other solutions are possible!

   Turning to the idea 4=3+1, divide the coins into three *big groups of four* coins and then each of these into a *small group of 3* and a *tiny group of 1* coin. Put two of the big groups on the scale and note the condition of the balance. If the scale is balanced then these two big groups have real coins and the fake one is in the third big group. If the scale is not balanced, then the coins in the third big group are all real. This is the first weighing.

   In the second weighing, rotate the small groups (of 3 coins), taking off the scale the one on the left pan, moving the small group from the right pan to the left, while placing the small group of the third big group on the right pan. Observe again the condition of the balance.

   If in the second weighing there is no change of the balance on the scale, i.e., it stays even or the same side is heavier, then all small coin groups are real coins and the fake one is in one of the three tiny groups (of 1 coin). Remove the small groups from the balance, and rotate the tiny groups (of 1 coin) as we did above for the small groups. This is the third weighing, and will identify the odd coin and determine its relative weight.

   If in the second weighing there is a change, it will identify the small group that contains the fake coin and in addition determine its relative weight. Put aside all other coins except the coins from the small group with the fake coin. Put one coin on each pan leaving the third one on the table. This is the third weighing in this case, which will identify the odd coin noting that the relative weight has already been determined.

5. Let A and B be two points in the plane. Describe the set S of all points in the plane such that for any point P in S we have $|PA| = 3|PB|$.

Solution: We are looking for all points P such that $|PA| = k|PB|$ where $k > 0$ is a given constant. Notice that for $k = 1$ this is just the line through the midpoint of AB which is perpendicular to AB. In the general, case the answer is a circle centered on the line determined by AB. The precise circle can be found in several ways.

The first solution we present uses Cartesian coordinates. Consider a coordinate system centered at A with x-axis pointing in the direction of B. The given points are $A(0, 0)$ and $B(c, 0)$ for some $c > 0$. A point $P(x, y)$ satisfies the wanted condition iff

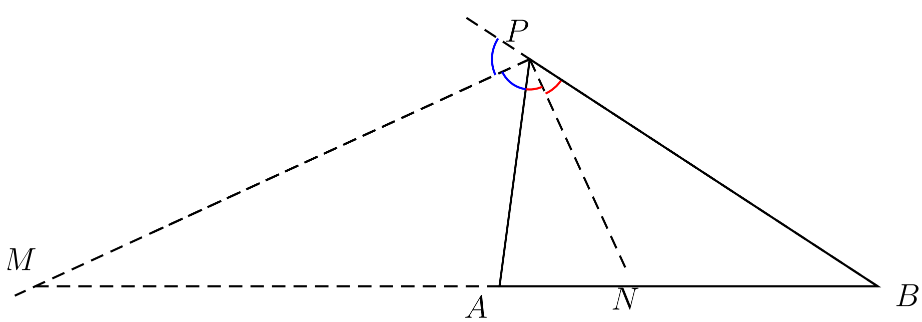
$$\frac{x^2 + y^2}{(x-c)^2 + y^2} = k^2.$$

Assuming that $k > 1$, the above equation can be written as follows by completing the square,

$$\begin{aligned} x^2 + y^2 &= k^2(x-c)^2 + k^2y^2 \\ (k^2-1)\left(x^2 - 2\frac{k^2c}{k^2-1} + \frac{k^2c^2}{k^2-1} + y^2\right) &= 0 \\ (k^2-1)\left(\left(x - \frac{k^2c}{k^2-1}\right)^2 + y^2\right) &= \frac{k^4c^2}{(k^2-1)^2} - \frac{k^2c^2}{k^2-1} \\ \left(x - \frac{k^2c}{k^2-1}\right)^2 + y^2 &= \frac{k^4c^2}{(k^2-1)^3} \end{aligned}$$

which is an equation of a circle centered at $(\frac{k^2c}{k^2-1}, 0)$ and radius $\frac{k^2c}{(k^2-1)\sqrt{k^2-1}}$. The case of $k < 1$ can be handles similarly.

Another solution can be obtained using Problem 3 and the similar identity for the outer angle.



Note that $\angle MPN = \pi/2$ while M and N are fixed on the line AB as determined by the ratio of the lengths $PA/PB = k$. Thus the location of all such P's is the circle with diameter MN.

6. A faulty calculator displayed $\diamond 38 \diamond 1625$ as an output of a calculation. We know that two of the digits of this number are missing and these are replaced with the symbol $\diamond$. Furthermore, we know that 9 and 11 divide the computed output. What are the missing digits and the complete output of our calculation?

Solution: Suppose the number is $\overline{x38y1625}$, x and y are digits. Then we have $9|x+y-2$ and $11|y-x+3-8-1+6-2+5$. Thus, $x+y-2=9k$ and $y-x+3=11n$ for some integers $k \geq 0$ and n. Hence, for the digits x and y we have

$$0 \leq y = 1/2(9k+11n) - 1/2 \leq 9 \qquad \text{and} \qquad 0 \leq x = 1/2(9k-11n) + 5/2 \leq 9.$$

Therefore, $0 \leq 9k+2 \leq 18$ which shows k equals 0 or 1. By inspection $k=1$, $n=0$ is the only solution, which shows that the number is 73841625.

7. Let A be the average of the three numbers $\sin 2\alpha$, $\sin 2\beta$ and $\sin 2\gamma$ where $\alpha+\beta+\gamma=\pi$. Express the product $P = \sin\alpha \sin\beta \sin\gamma$ in terms of A.

Solution: From the given condition and a small calculation we have [1]

$$\begin{aligned}
-1 &= e^{i\pi} = e^{i(\alpha+\beta+\gamma)} = e^{i\alpha}e^{i\beta}e^{i\gamma} \\
&= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma) \\
&= \cos\alpha\cos\beta\cos\gamma - \cos\alpha\sin\beta\sin\gamma - \cos\beta\sin\alpha\sin\gamma - \cos\gamma\sin\alpha\sin\beta \\
&\quad + i(\cos\alpha\cos\beta\sin\gamma + \cos\beta\cos\gamma\sin\alpha + \cos\alpha\cos\gamma\sin\beta - \sin\alpha\sin\beta\sin\gamma).
\end{aligned}$$

In particular, the imaginary part of the right-hand side vanishes, i.e.,

$$\sin\alpha\sin\beta\sin\gamma = \cos\alpha\cos\beta\sin\gamma + \cos\beta\cos\gamma\sin\alpha + \cos\alpha\cos\gamma\sin\beta. \tag{1}$$

Next, we will use that

$$\cos\alpha\sin\gamma + \cos\gamma\sin\alpha = \sin(\alpha+\gamma), \tag{2}$$

which follows easily from a comparison of the imaginary parts of

$$(\cos\alpha + i\sin\alpha)(\cos\gamma + i\sin\gamma) = e^{i\alpha}e^{i\gamma} = e^{i(\alpha+\gamma)} = \cos(\alpha+\gamma) + i\sin(\alpha+\gamma).$$

Therefore, we have

$$\begin{aligned}
&\cos\alpha\cos\beta\sin\gamma + \cos\beta\cos\gamma\sin\alpha = \cos\beta(\cos\alpha\sin\gamma + \cos\gamma\sin\alpha) \\
&\qquad = \cos\beta\sin(\alpha+\gamma) = \cos\beta\sin(\pi-\beta) = \cos\beta\sin\beta = \frac{1}{2}\sin 2\beta,
\end{aligned}$$

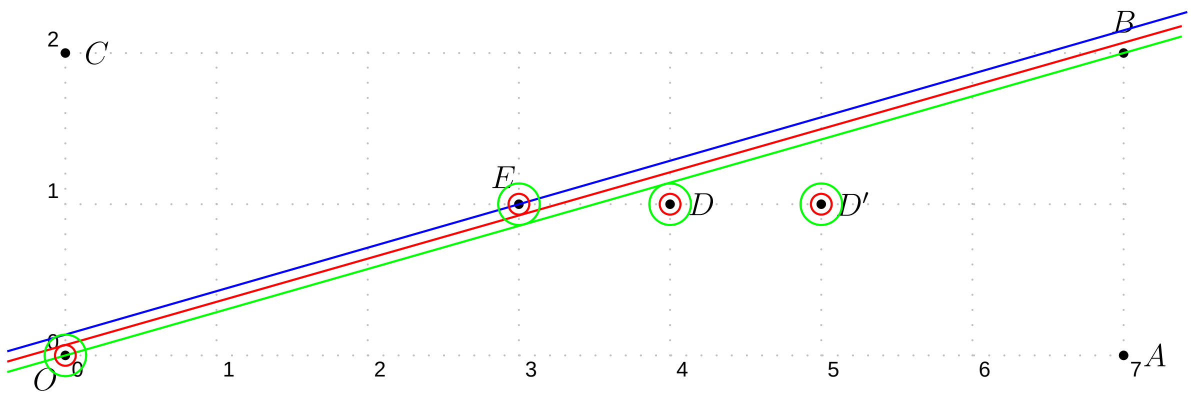
since $2\cos\beta\sin\beta = \sin 2\beta$ by (2). Finally, from the above we can rewrite (1) as

$$\begin{aligned}
&\sin\alpha\sin\beta\sin\gamma = \cos\alpha\cos\beta\sin\gamma + \cos\beta\cos\gamma\sin\alpha + \cos\alpha\cos\gamma\sin\beta \\
&\quad = \frac{1}{2}(\cos\alpha\cos\beta\sin\gamma + \cos\beta\cos\gamma\sin\alpha) + \frac{1}{2}(\cos\beta\cos\gamma\sin\alpha + \cos\alpha\cos\gamma\sin\beta) \\
&\qquad + \frac{1}{2}(\cos\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma\sin\beta) = \frac{1}{4}(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = \frac{3}{4}A.
\end{aligned}$$

8. Suppose we draw circles of radius r with centers at every point in the plane with integer coordinates. What is the smallest r such that every line with slope $2/7$ has a point in common with at least one of these circles?

Solution: Fix a Cartesian coordinate system in the plane. We will call the points with integer coordinates lattice points. Consider the line l with slope $2/7$ through the origin, i.e., the green line through the points O and B below.

[1] We only need the imaginary part in the above identity so the calculation of the real part is given only for completeness, but is unnecessary.



We want to find an $r > 0$ as small as possible so that for any parallel to l line there is some circle of radius r with center at a lattice point which a point in common with this line. Take a point on the lattice which is not on l and is as close to l as possible (without being on l). There is such a point since l has a rational slope so the rectangle $OABC$ "repeats" along the length of l (the picture is "periodic" under translations by the vector $(7,2)$), hence it is enough to compare the (finitely many) distances to the line l from the lattice points in the rectangle $OABC$. In the picture above E and D are such points.

The "green" circles are centered at the lattice points and have radii equal to the distance between E and l, while the "red" circles are half the size of the "green" circles. Notice that if we draw identical circles of even smaller radii than the red circles, then we will be able to put a line parallel to l which has no common points with any of these circles as there will be an "opening" between them. On the other hand taking larger radii will cause an intersection between any line and the family of circles since it occurs already for the "red" circles. Thus, the answer is half of the distance between the point E and the line l.

For the exact numerical values, first we need to determine the smallest non-zero distance between the lattice points and the line l. For this we recall that given a line $l:\ ax+by+c=0$ and a point $P(x_0,y_0)$ then the distance between P and the line is $proj_{\vec{n}}\overrightarrow{OP}$ where $\vec{n}$ is a vector normal to the line, for example $\vec{n}=(a,b)$. In other words

$$dist(P,l)=\frac{\left|\vec{n}\cdot\overrightarrow{OP}\right|}{|\vec{n}|}.$$

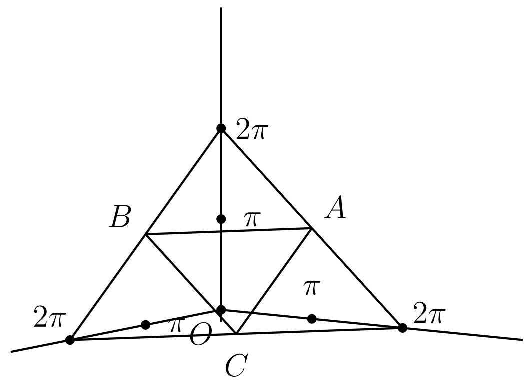
By the periodicity, it is enough to compare the distances between the lattice points in the upper half of the rectangle $OABC$ and the line. This is how the point E is determined. Finally,

$$d(E,l)=\left|\frac{(2,-7)\cdot(3,1)}{\sqrt{2^2+7^2}}\right|=1/\sqrt{53}.$$

Therefore, the numerical answer is $\frac{1}{2\sqrt{53}}$.

9. What is the probability of picking at random three points on a circle of radius one so that all three lie in a semicircle?

Solution: For three points on the unit circle which are the vertices of a triangle we will determine the probability a that the triangle is acute, hence the sought probability is $o=1-a$.



A triangle on the unit circle is determined by the lengths α, β and γ of the corresponding arcs. Thus $\alpha+\beta+\gamma=2\pi$. Such triples can be represented as the points inside the triangle with vertices $A'(2\pi,0,0)$, $B'(0,2\pi,0)$ and $C'(0,0,2\pi)$ in the Euclidean three space. The triangle on the circle is acute iff each of the given arcs is less than π. Thus, the configurations leading to an acute triangle are those satisfying the system

$$\alpha+\beta+\gamma=2\pi, \qquad \alpha,\ \beta,\ \gamma>0,$$
$$\alpha<\pi, \quad \beta<\pi, \quad \gamma<\pi.$$

Geometrically, these are the points from the "middle" triangle $\triangle ABC$ whose vertices are at the midpoints of the sides of $\triangle A'B'C'$. Therefore, the probability for picking an acute triangle is $1/4$ while the probability for an obtuse triangle is $o=3/4$.

10. Solve $\underbrace{\overline{aa...a}}_{2k}-\underbrace{\overline{bb...b}}_{k}=\left(\underbrace{\overline{cc...c}}_{k}\right)^2$, where $\underbrace{\overline{cc...c}}_{k}$ denotes a number with k digits each one equal to c.

Solution: Since

$$\underbrace{\overline{cc...c}}_{k}=\left(1+10+10^2+...+10^{k-1}\right)\cdot c=\frac{10^k-1}{9}\cdot c$$

we need to solve for $a,b,c\in\{1,2,...,9\}$ and $k-$positive integer the equation

$$\begin{aligned}\frac{10^{2k}-1}{9}\cdot a-\frac{10^k-1}{9}\cdot b &= \left(\frac{10^k-1}{9}\right)^2\cdot c^2\\ 9\left(10^{2k}-1\right)a-9\left(10^k-1\right)b &= \left(10^k-1\right)^2c^2\\ 9\left(10^k+1\right)a-9b &= \left(10^k-1\right)c^2\\ \left(9a-c^2\right)10^k &= 9\,(b-a)-c^2.\end{aligned}$$

From the given condition we have $9\,(b-a)-c^2\leq 9\,(9-1)-1=71$ and $9\,(b-a)-c^2\geqq 9\,(1-9)-81=-153$. while the left-hand side is divisible by 10. Thus a possible solution satisfies

$$9\,(b-a)-c^2=10n \quad\text{and}\quad \left(9a-c^2\right)10^{k-1}=n,$$

where $n=-10,-9,-8,0,\pm1,\pm2,..,\pm7$.

If we consider the second equation, we notice that for $k\geqq 2$ it follows 10 divides n, hence $n=0$ or $n=-10$. Thus, for $k\geqq 2$ we have either $k=2$ and

$$9\,(b-a)-c^2=-100$$
$$9a-c^2=-1$$

or $k \geqq 2$ and

$$\begin{gathered} 9\,(b-a) - c^2 = 0 \\ 9a - c^2 = 0. \end{gathered}$$

The general solution of the first system (using a as a parameter) is

$$\left[b = 2a - 11, c = \sqrt{9a+1}\right], \left[b = 2a - 11, c = -\sqrt{9a+1}\right].$$

Thus, the solution of the original problem is

$$a = 7, b = 3, c = 8, k = 2, \text{ i.e., } 7777 - 33 = 88^2$$

The general solution of the second system (using a as a parameter) is:

$$\left[b = 2a, c = 3\sqrt{a}\right], \left[b = 2a, c = -3\sqrt{a}\right].$$

Thus, the solution of our problem for any $k \geqq 2$ are the triples

$$a = 1, b = 2, c = 3 \quad \text{and} \quad a = 4, b = 8, c = 6.$$

These correspond to $\underbrace{\overline{11...1}}_{2k} - \underbrace{\overline{22...2}}_{k} = \left(\underbrace{\overline{33...3}}_{k}\right)^2$ and $\underbrace{\overline{44...4}}_{2k} - \underbrace{\overline{88...8}}_{k} = \left(\underbrace{\overline{66...6}}_{k}\right)^2$.

Let us consider the case $k = 1$. By a direct inspection the solutions here are:

$$11 - 2 = 3^2,\ 11 - 7 = 2^2,\ 22 - 6 = 4^2,\ 33 - 8 = 5^2,$$

$$44 - 8 = 6^2,\ 55 - 6 = 7^2,\ 66 - 2 = 8^2,\ 88 - 7 = 9^2.$$

Overall, we showed that all solutions of the given problem are

$$7777 - 33 = 88^2,$$

$$\underbrace{\overline{11...1}}_{2k} - \underbrace{\overline{22...2}}_{k} = \left(\underbrace{\overline{33...3}}_{k}\right)^2, \ \underbrace{\overline{44...4}}_{2k} - \underbrace{\overline{88...8}}_{k} = \left(\underbrace{\overline{66...6}}_{k}\right)^2, \ k \geq 1,$$

$$11 - 7 = 2^2,\ 22 - 6 = 4^2,\ 33 - 8 = 5^2,$$

$$55 - 6 = 7^2,\ 66 - 2 = 8^2,\ 88 - 7 = 9^2.$$