UNM - PNM STATEWIDE MATHEMATICS CONTEST XLVI - SOLUTIONS

February 1, 2014 Second Round Three Hours

1. Four siblings BRYAN, BARRY, SARAH and SHANA are having their names monogrammed on their towels. Different letters may cost different amounts to monogram. If it cost \$21 to monogram BRYAN, \$25 to monogram BARRY and \$18 to monogram SARAH, how much does it cost to monogram SHANA?

   Answer: \$14.

   Solution: Notice that BRYAN, SARAH is an anagram of BARRY, SHANA. Since the price to monogram BRYAN and SARAH should be equal to the price to monogram BARRY and SHANA, then $21 + 18 = 25 + x$ where x is the price to monogram SHANA. So $x = 14$.

   Alternatively, if we denote with the respective letters the amounts they cost, we have

$$\begin{aligned} B+R+Y+A+N &= 21 \\ B+A+2R+Y &= 25 \\ S+A+R+A+H &= 18 \end{aligned}$$

   i.e,

$$\begin{aligned} A+B\ +N+R\ +Y &= 21 \\ A+B\quad +2R\ +Y &= 25 \\ 2A\quad +H\ +R+S\ &= 18 \end{aligned}$$

   The question is to find $2A + H + N + S$. If we add the first and the third equations in the system and subtract the second equation from the result we obtain $2A + H + N + S = 21 + 18 - 25 = 14$.

2. If $f(x) = x^3 + 6x^2 + 12x + 6$, solve the equation $f(f(f(x))) = 0$.

   Answer: Note that $f(x) = (x + 2)^3 - 2$. Thus $f(f(x)) = ((x + 2)^3 - 2 + 2)^3 - 2 = (x + 2)^9 - 2$ and $f(f(f(x))) = \left((x + 2)^9 - 2 + 2\right)^3 - 2 = (x + 2)^{27} - 2$. So $f(f(f(x))) = 0$ has solution $-2 + \sqrt[27]{2}$.

3. Two people, call them A and B, are having a discussion about the ages of B's children.
   A: "What are the ages, in years only, of your four children?"
   B: "The product of their ages is 72."
   A: "Not enough information."
   B: "The sum of their ages equals your eldest daughter's age."
   A: "Still not enough information."
   B: "My oldest child who is at least a year older than her siblings took the AMC 8 for the first time this year."
   A: "Still not enough information."
   B: "My youngest child is my only son."
   A: "Now I know their ages.."
   What are their ages?

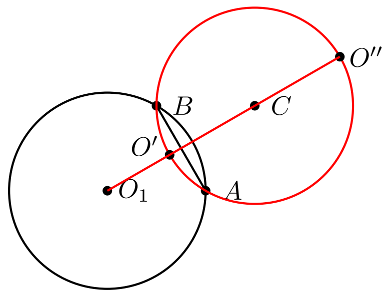
   Answer: Their ages are 1, 3, 3 and 8. Let the ages of the children be a, b, c, d. Then $abcd = 72$ by the answer to the first question. The possibilities for the quadruple a, b, c, d along with their sum (in parenthesis) are:
   i) 1, 1, 1, 72 (75); ii) 1, 1, 2, 36 (40); iii) 1, 1, 3, 24 (29); iv) 1, 1, 4, 18 (24); v) 1, 1, 6, 12 (20); vi) 1, 1, 8, 9 (19); vii) 1, 2, 2, 18 (23); viii) 1, 2, 3, 12 (18); ix) 1, 2, 4, 9 (16); x) 1, 2, 6, 6 (15); xi) 1, 3, 3, 8 (15); xii) 1, 3, 4, 6 (14); xiii) 2, 2, 2, 9 (15); xiv) 2, 2, 3, 6 (13); xv) 2, 3, 3, 4 (12);

   Since A cannot determine the ages after computing the sum then their ages must sum to 15 since this is the only number which is repeated. There is a single oldest child so this rules out the ages 1, 2, 6, 6 and there is a single youngest child which rules out 2, 2, 2, 9. So the ages must be 1, 3, 3, 8.

4. Find the smallest and largest possible distances between the centers of two circles of radius 1 such that there is an equilateral triangle of side of length 1 with two vertices on one of the circles and the third vertex on the second circle.

   Solution: If O_1 and O_2 are the two centers then $\sqrt{3}-1 \le |O_1O_2| \le \sqrt{3}+1$ and the smallest and largest possible distances between the two centers are $\sqrt{3}-1$ and $\sqrt{3}+1$ respectively. In order to prove this we argue as follows. Let AB be a chord of length 1 in a circle ("the first circle") of radius 1. Consider an equilateral triangle with one side AB. The third vertex is either the center O_1 of the first circle or a point C outside of the first circle.

The center of the second circle can be anywhere on a circle of radius 1 centred at the third vertex. If the third vertex is O_1 then the center of the second circle lies on the first circle hence the distance between the two centers is 1. If the third vertex is the point C then the center of the second circle can be anywhere on the red circle on the figure below. In this case, the intersection of the red circle with the line O_1C gives the closest and the most distant positions between the two centers. It is enough to show that $|O_1C| = \sqrt{3}$ since $\sqrt{3}-1 < 1 < \sqrt{3}+1$, $|O_1O'| = \sqrt{3}-1$ and $|O_1O'| = \sqrt{3}+1$. In order to compute the respective distances between the two centers in the extremal configurations we can use, for example, that the length of O_1C is twice the height of the equilateral triangle $\triangle ABC$.



5. 5^n is written on the blackboard. The sum of its digits is calculated. Then the sum of the digits of the result is calculated and so on until we have a single digit. If $n = 2014$, what is this digit?

Answer: 4

Solution: Let $S(m)$ be the digit representing the iterated sum described above - this is frequently called the "digital root" function. To gain an idea of how to proceed we compute a few "digital roots"

$$S(5^1) = 5, \quad S(5^2) = S(25) = 2+5 = 7, \quad S(5^3) = S(125) = 1+2+5 = 8,$$
$$S(5^4) = S(625) = S(13) = 4, \quad S(5^5) = S(3+1+2+5) = S(11) = 2,$$
$$S(5^6) = S(15625) = S(19) = S(10) = 1,$$

$$S(5^7) = S(78125) = S(23) = 5,$$
$$S(5^8) = S(390625) = S(25) = 7, \quad S(5^9) = S(1953125) = 8$$
$$S(5^{10}) = S(9765625) = 4, \quad S(5^{11}) = S(48828125) = 2, \quad S(5^{12}) = S(244140625) = 1,$$

$$S(5^{13}) = S(1220703125) = 5, \ldots.$$

The above calculations suggest that $S(5^n)$ is determined by the remainder of n when divided by 6, $i.e., S(5^n)$ is determined by the remainder of the power modulo 6. If this is the case, then since $2014 = 6 * 335 + 4$, i.e., $2014 \equiv 4 \mod 6$ we have $S(5^{2014}) = 4$.

One way to justify and give a rigourous solution of the problem is to use the fact that if $r = (m \mod 9)$, i.e., $0 \le r \le 8$ is the remainder of m when divided by 9, then

$$S(m) = \begin{cases} r, & r \neq 0; \\ 9, & r = 0. \end{cases}$$

Indeed, consider $\sigma(m)$ which for a positive integer number m gives the sum of the digits of m. Thus $S(m)$ is computed by applying σ several times until we obtain a single digit number (and further applications of σ give the same number). Since $m \equiv \sigma(m) \mod 9$ for any integer m it follows $m \equiv S(m) \mod 9$. Taking into account that the $1 \le S(m) \le 9$ it follows that $S(m) = (m \mod 9)$ unless $9|m$ in which case $S(m) = 9$. The above implies that for any positive integers m and k we have $S(m+k) = S\left(S(m) + S(k)\right)$ hence, since a product is a multiple times addition, also

(1) $$S(m * k) = S\left(S\left(m\right) * S\left(k\right)\right).$$

The formula (1) implies $S(5^n) = S\left(5 * S(5^{n-1})\right)$. Thus we have

$$S(5^1) = 5, \quad S(5^2) = S(25) = 7, \quad S(5^3) = S(5*7) = 8, \quad S(5^4) = S(5*8) = 4$$
$$S(5^5) = S(5*4) = 2, \quad S(5^6) = S(5*2) = 1,$$

$$S(5^7) = S(5*1) = 5, \quad S(5^8) = S(5*5) = 7, \quad S(5^9) = S(5*7) = 8, \quad S(5^{10}) = S(5*8) = 4,$$
$$S(5^{11}) = S(5*4) = 2, \quad S(5^{12}) = S(5*2) = 1,$$

$$S(5^{13}) = S(5*1) = 5, \ldots.$$

By induction on n ($5^n \mod 9$ is determined by $n \mod 6$) it follows $S(5^{2014}) = 4$.

6. How many triples (x, y, z) of rational numbers satisfy the following system of equations?

$$x + y + z = 0$$
$$xyz + 4z = 0$$
$$xy + xz + yz + 2y = 0$$

Answer: 2 solutions: $(0, 0, 0)$ and $(-2, 2, 0)$ are the only triples of rational numbers which satisfy the system. Considering the second equation above, we can conclude that $z = 0$ or $xy + 4 = 0$.

In the case that $z = 0$, the first equation implies that $x = -y$ and the third equation implies that $xy + 2y = 0$. Substituting $x = -y$ into this equation we obtain that $2y - y^2 = 0$ implying that $y = 0$ or $y = 2$. Thus $(0, 0, 0)$ and $(-2, 2, 0)$ are solutions.

Now assume that $z \neq 0$. So $xy = -4$. Plugging this into the third equation, we obtain

$$0 = -4 + xz + yz + 2y = -4 - (y + z)z + yz + 2y = -4 - z^2 + 2y$$

or $y = \frac{1}{2}z^2 + 2$. Since $x = -y - z$, we can plug this into the third equation and use the fact that $y - 2 = \frac{1}{2}z^2$ and obtain,

$$0 = -(y + z)^2 + yz + 2y = -y^2 - 2yz - z^2 + yz + 2y = -y^2 - yz - z^2 + 2y$$
$$= -yz - z^2 + y(2 - y) = -yz - z^2 - \frac{1}{2}yz^2$$

Since $z \neq 0$, we can divide by $-z$ and we get $0 = y + z + \frac{1}{2}yz$. Again substitute $y = \frac{1}{2}z^2 + 2$ and we get the following cubic $\frac{1}{4}z^3 + \frac{1}{2}z^2 + 2z + 2 = 0$ or

$$z^3 + 2z^2 + 8z + 8 = 0.$$

By the "rational root theorem", whose result we prove below, if this cubic has rational roots, they are integers and must be among $\pm 1, \pm 2, \pm 4$. None of these are solutions so we have no solutions when $z \neq 0$. Rather than proving the general rational root theorem, we consider the special case of $z^3 + 2z^2 + 8z + 8 = 0$. If $z = m/n$ is a rational root with m and n relatively prime integers (m and n have no common divisor other than ± 1), $(m, n) = 1$, then

$$m^3 + 2m^2n + 8mn^2 + 8n^3 = 0,$$

hence $-8n^3 = m^3 + 2m^2n + 8mn^2 = m(m^2 + 2mn + 8n^2)$ and $-m^3 = 2m^2n + 8mn^2 + 8n^3 = n(2m^2 + 8mn + 8n^2)$. Since $(m, n) = 1$, the first equation shows that $m|8$ while the second equation gives $n|1$, i.e., $n = \pm 1$.

7. Let k be a natural number. Show that the sum of the k-th powers of the first n positive integers is a polynomial of degree $k + 1$, i.e.,

$$1^k + 2^k + 3^k + \cdots + n^k = p_{k+1}(n),$$

where $p_{k+1}(t)$ is a polynomial of degree $k + 1$. For example, for $k = 1$ we have

$$1 + 2 + \cdots + n \equiv \sum_{j=1}^{n} j = \frac{n(n+1)}{2} = 1/2\, n^2 + 1/2\, n,$$

hence $p_2(t) = 1/2\, t^2 + 1/2\, t$.

Solution: We shall use induction on k and the binomial formula

$$(a + b)^l = \sum_{k=0}^{l} C_{l,k}\ a^{l-k} b^k \equiv C_{l,0} a^l + C_{l,1} a^{l-1} b + \cdots + C_{l,l}\ b^l,$$

where l is any positive integer and $C_{l,k}$, denoted usually by $\binom{l}{k}$, are certain constants with $C_{l,l} = 1$. The exact values of the remaining binomial coefficients $C_{l,k}$'s will not matter except that they are non-zero. In fact, we know that $C_{l,0} = 1$ and

$$C_{l,k} \equiv \binom{l}{k} = \frac{l!}{(l-k)!k!}$$

otherwise.

When $k = 0$ the claim is true since $1^0 + 2^0 + \ldots n^0 = n$ hence $p_1(t) = t$ is the sought polynomial.

To get some insight, it is helpful to derive the case $k = 1$ with $p_2(t) = 1/2\, t^2 + 1/2\, t$ using the case $k = 0$ (other proofs of the case $k = 1$ are possible!). Consider $(a+1)^2 - a^2 = 2a$, which we write for $a = 1, 2, \ldots, n$ and then sum the n equations. The result is, after cancelling the terms in the left hand side,

$$(n+1)^2 - 1^2 = 2\sum_{a=1}^{n} a.$$

This shows that for any positive integer n we have

$$\sum_{a=1}^{n} a = \frac{1}{2}\left((n+1)^2 - 1\right) = \frac{1}{2}n(n+1).$$

Turning to the induction step, let m be a positive integer. Suppose that we know the claim for $k = 1, 2, \ldots, m-1$. We will show that the claim is true for $k = m$. From the binomial formula we have

$$(1+b)^{m+1} - b^{m+1} = \sum_{k=0}^{m} C_{m+1,k}\ b^k$$

since $C_{m+1,m+1} = 1$. If we write this equation for $b = 1, 2, \ldots, n$ we obtain the next n equations

$$\begin{aligned}
2^{m+1} - 1^{m+1} &= \sum_{k=0}^{m} C_{m+1,k}\ 1^k \\
3^{m+1} - 2^{m+1} &= \sum_{k=0}^{m} C_{m+1,k}\ 2^k \\
4^{m+1} - 3^{m+1} &= \sum_{k=0}^{m} C_{m+1,k}\ 3^k \\
&\cdots\cdots \\
(1+b)^{m+1} - b^{m+1} &= \sum_{k=0}^{m} C_{m+1,k}\ b^k \\
&\cdots\cdots \\
(n+1)^{m+1} - n^{m+1} &= \sum_{k=0}^{m} C_{m+1,k}\ n^k.
\end{aligned}$$

Now, we sum the above n-equations noticing that in the left hand side the terms cancel except two of them while in the right hand side we reorder the terms in powers of k, which gives

$$(n+1)^{m+1} - 1^m = \sum_{b=1}^{n}\Big(\sum_{k=0}^{m} C_{m+1,k}\ b^k\Big) = \sum_{k=0}^{m} C_{m+1,k}\Big(\sum_{b=1}^{n} b^k\Big).$$

By our assumptions, for $k = 0, 1, \ldots, m-1$ we know that $\sum_{b=1}^{n} b^k = p_k(n)$ for certain polynomials p_{k+1} such that p_{k+1} is of degree $k+1$. Therefore we have

$$(n+1)^{m+1} - 1^m = \sum_{b=1}^{n}\Big(\sum_{k=0}^{m} C_{m+1,k}\ b^k\Big) = C_{m+1,m}\Big(\sum_{b=1}^{n} b^m\Big) + \sum_{k=0}^{m-1} C_{m+1,k}\ p_{k+1}(n).$$

Letting

$$p(t) = \sum_{k=0}^{m-1} C_{m+1,k}\ p_{k+1}(t)$$

it follows that $p(t)$ is a polynomial of degree m. Taking into account that $C_{m+1,m} = m+1 \neq 0$ we have

$$\sum_{b=1}^{n} b^m = \frac{1}{m+1}\Big((n+1)^{m+1} - 1 - p(n)\Big).$$

Thus, $p_{m+1}(t) = \frac{1}{m+1}\Big((n+1)^{m+1} - 1 - p(n)\Big)$ is a polynomial of degree $m+1$ and

$$\sum_{b=1}^{n} b^m = p_{m+1}(n),$$

which is what we needed to show.

8. A certain country uses bills of denominations equivalent to $15 and $44. The ATM machines in this country can give at a single withdrawer any amount you request as long as both bills are used. Show that you can withdraw $x if and only if you cannot withdraw $y, where $x + y = 719$.

   Answer: As an example, consider the amount $15 * 44 = 660$. This is an amount that actually cannot be withdrawn. Otherwise, 15*44=15m+44n for some positive integers m and n which is a contradiction since then $15|44n$ (i.e., 15 divides 44n) and $44|15m$ hence $15|n$ and $44|m$, hence $15m+44n > 15*44$. On the other hand, the smallest amount we can withdraw is 15+44=59. Notice that, $59 + 660 = 719$ so the claim is true in this particular case.

   In the next step we show that any amount larger than $660 can be withdrawn. For this we use that 15, $15*2$, $15*3$,..., $15*44$ is a complete set of residues mod 44. Indeed, clearly 44 cannot divide the difference of any two of them hence the remainders of the numbers $15m$, $m = 1, 2, \ldots, 44$ when divided by 44 are different. On the other hand we have exactly 44 numbers so every remainder appears exactly once. Therefore, for any integer $x > 15*44 = 660$ we can find an integer m, $1 \le m \le 44$ such that $44|15m$. This shows that x can be written as $x = 15m + 44n$ for some positive integers m and n.

   We are left with case $59 < x < 660$. Suppose $x = 15m + 44n$ and $y = 719 - x = 15a + 44b$ for some positive integers m, n, a and b. Then we have $15(a+m)+44(b+n) = 719$ or $15(a+m-1)+44(b+n-1) = 660$ which is a contradiction with the fact that 660 cannot be written as a linear combination of 15 and 44 with positive integers (660 cannot be withdrawn).

9. Suppose that f is a mapping of the plane into itself such that the vertices of every equilateral triangle of side one are mapped onto the vertices of a congruent triangle. Show that the the map f is distance preserving, i.e., $d(p,q) = d(f(p), f(q))$ for all points p and q in the plane, where $d(x,y)$ denotes the distance between the points x and y in the plane. In other words, if any two points that are 1 unit apart are mapped to points that are one unit apart, then any two points are mapped to two points that are the same distance as their pre-images.

   Solution: The proof is contained in the paper, *On isometries of Euclidean spaces*, F. S. Beckman and D. A. Quarles. Proc. Amer. Math. Soc. 4 (1953), 810–815. Full-text PDF on the AMS journal web site.

   The proof has several steps. First, consider a rhombus formed by two equilateral triangles of sides of length one. The vertices of such a rhombus are mapped to the vertices of a congruent rhombus since the two ends of the "long" diagonal (of length $\sqrt{3}$) cannot be mapped to the same point by Problem 4. In particular, the distance $\sqrt{3}$ is preserved.

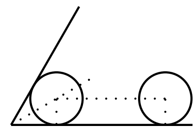
   If we consider then a regular hexagon of side of length 1 we see that the distance 2 is preserved. By "adding" more hexagons we can see that any integer distance is preserved.

   Then we show that distances $A/2^B$ are preserved for any positive integers A and B. For this we consider an isosceles triangle with two sides of length $2A$ and the third of length A and show that the distance $A/2$ is preserved, and then iterate the already established results.

   Using a density argument (between any two real numbers a and b there is a number of the type $A/2^B$ between them) we see that any distance is preserved.

10. Given a sheet in the shape of a rhombus whose side is 2 meters long and one of its angles is 60^o what is the maximum area that can be cut out of the sheet if we are allowed to cut two discs.

    Solution: First we will argue that if two discs can be cut from the rhombus then we can cut them by positioning both centers on one of the diagonal of the rhombus. A useful fact that will be used in the argument is that given an acute angle and a disc inside the region bounded by it, then there is a congruent disc lying also completely inside the angle with center on the bisector which is "closest" to the vertex - push the center parallel to the closest side until the center reaches the diagonal.



Turning to the stated claim, we consider the two possible cases: (i) both centers are on the short diagonal of the rhombus; and (ii) the centers are not on the short diagonal. In the second case we can push the centers to the long diagonal keeping the discs non-overlapping. In fact, if one of the centers is closer to an end of the long diagonal than the other center, then we can "push" the "closer" center to the long diagonal as close to the closest vertex as possible while positioning the center of the other disc to the opposite end of the long diagonal. In the case when the centers are symmetric with respect to the long diagonal, so the closest end of the long diagonal is the same for both centers, we can use a reflection with respect to the short diagonal to see that we can position the centers as claimed.

At this point, we have reduced the problem to the case when both centers are on one of the diagonals. It is not hard to see that it is enough to consider the case when both centers are on the long diagonal. However, since the analysis of finding the maximum area we can cut is very similar in both cases we consider first the configuration when both centers are on the long diagonal. Let the radii of the discs be R and r, respectively, with $R \geq r$. Since we want to maximize the area we can assume that the discs are as big as possible, i.e., they touch each other and the sides of the rhombus. There are two "extreme" configurations - the two disks are congruent, $R = r$, and the other is when R is as large as possible, i.e., one of the discs is inscribed in the rhombus and the other is inscribed in the region bounded by one of the acute angles of the rhombus and the inscribed circle. We will prove that the latter configuration gives is the one with maximum combined area, i.e., $A = \pi(R^2 + r^2)$ is as large as possible. Using that in a right triangle with the remaining angles equal to 30^o and 60^o the hypotenuse is twice the side across the 30^o angle we see that $3R + 3r = 2\sqrt{3}$-the length of the long diagonal. Furthermore, as determined by the two "extreme" configurations, we have that R is greater than or equal to the radius of the inscribed circle in an equilateral triangle of side 2 and less than or equal to the radius of the circle inscribed in the rhombus. Thus,

$$\frac{1}{\sqrt{3}} \equiv \sqrt{3}/3 \ \leq R \leq \ \sqrt{3}/2 = \frac{3}{2\sqrt{3}}.$$

Using the relation between R and r we can write

$$\frac{1}{\pi}A = R^2 + \left(\frac{2}{\sqrt{3}} - R\right)^2 = 2\left(\frac{1}{\sqrt{3}} - R\right)^2 + \frac{2}{3}.$$

Thus the cut area is maximized in this case when we take R as far away from $\frac{1}{\sqrt{3}}$ as possible, i.e, $R = \sqrt{3}/2$ which corresponds to cutting the disc inscribed in the rhombus and the smaller disc inscribed in the region bounded by one of the acute angles of the rhombus and the inscribed circle. This gives a total area of

$$A = 2\pi\left(\frac{1}{\sqrt{3}} - \frac{3}{2\sqrt{3}}\right)^2 + \pi\frac{2}{3} = 5\pi/6.$$

Working similarly in the case when both centers are on the shorter diagonal we can see that the maximum area we can cut is less than the area when both centers are on the long diagonal.