UNM - PNM STATEWIDE MATHEMATICS CONTEST XLVI

February 1, 2014 Second Round Three Hours

1. Four siblings BRYAN, BARRY, SARAH and SHANA are having their names monogrammed on their towels. Different letters may cost different amounts to monogram. If it cost \$21 to monogram BRYAN, \$25 to monogram BARRY and \$18 to monogram SARAH, how much does it cost to monogram SHANA?
2. If $f(x) = x^3 + 6x^2 + 12x + 6$, solve the equation $f(f(f(x))) = 0$.
3. Two people, call them A and B, are having a discussion about the ages of B's children.
   A: "What are the ages, in years only, of your four children?"
   B: "The product of their ages is 72."
   A: "Not enough information."
   B: "The sum of their ages equals your eldest daughter's age."
   A: "Still not enough information."
   B: "My oldest child who is at least a year older than her siblings took the AMC 8 for the first time this year."
   A: "Still not enough information."
   B: "My youngest child is my only son."
   A: "Now I know their ages.."
   What are their ages?
4. Find the smallest and largest possible distances between the centers of two circles of radius 1 such that there is an equilateral triangle of side of length 1 with two vertices on one of the circles and the third vertex on the second circle.
5. 5^n is written on the blackboard. The sum of its digits is calculated. Then the sum of the digits of the result is calculated and so on until we have a single digit. If $n = 2014$, what is this digit?
6. How many triples (x, y, z) of rational numbers satisfy the following system of equations?

$$x + y + z = 0$$

$$xyz + 4z = 0$$

$$xy + xz + yz + 2y = 0$$

7. Let k be a natural number. Show that the sum of the k-th powers of the first n positive integers is a polynomial of degree $k + 1$, i.e.,

$$1^k + 2^k + 3^k + \cdots + n^k = p_{k+1}(n),$$

   where $p_{k+1}(t)$ is a polynomial of degree $k + 1$. For example, for $k = 1$ we have

$$1 + 2 + \cdots + n \equiv \sum_{j=1}^{n} j = \frac{n(n+1)}{2} = 1/2\, n^2 + 1/2\, n,$$

   hence $p_2(t) = 1/2\, t^2 + 1/2\, t$.
8. A certain country uses bills of denominations equivalent to \$15 and \$44. The ATM machines in this country can give at a single withdrawer any amount you request as long as both bills are used. Show that you can withdraw \$x if and only if you cannot withdraw \$y, where $x + y = 719$.

9. Suppose that f is a mapping of the plane into itself such that the vertices of every equilateral triangle of side one are mapped onto the vertices of a congruent triangle. Show that the the map f is distance preserving, i.e., $d(p, q) = d(f(p), f(q))$ for all points p and q in the plane, where $d(x, y)$ denotes the distance between the points x and y in the plane. In other words, if any two points that are 1 unit apart are mapped to points that are one unit apart, then any two points are mapped to two points that are the same distance as their pre-images.
10. Given a sheet in the shape of a rhombus whose side is 2 meters long and one of its angles is 60^o what is the maximum area that can be cut out of the sheet if we are allowed to cut two discs.