UNM - PNM STATEWIDE MATHEMATICS CONTEST XLIV

November 4-7, 2011 First Round Three Hours

1. How many positive integer numbers less than or equal to 2011 are multiples of both 3 and 5 but not multiples of 8?

   Answer: 118. Solution: Let D_k be the set of positive integers less than or equal to 2011 which are *multiples of* (we say also *divisible by*) k. Let $|D_k|$ be the number of elements of the set D_k, i.e., the number of positive integers less than or equal to 2011 which are multiples of k. With this notation, the question is to find the number $|D_3 \cap D_5| - |D_3 \cap D_5 \cap D_8|$, where the symbol $\cap$ is used, as usual, for the intersection (the common elements) of the involved sets. Thus, for example, $D_3 \cap D_5$ is the set of positive integers less than or equal to 2011 which are multiples of both 3 and 5. A key observation, which we shall used next, is that if m and n are two relatively prime integer numbers, i.e., they do not have any common integer divisor besides ± 1, then any number which is divisible by both m and n is also divisible by mn. Therefore.

   $$|D_3 \cap D_5| - |D_3 \cap D_5 \cap D_8| = |D_{15}| - |D_{120}|.$$

   Next, we use that $|D_k|$ is the number of times k "fits" inside 2011. In other words, $|D_k|$ is the largest positive number whose product with k is less than or equal to 2011, but whose product with $k+1$ is greater than 2011. The "floor" function is defined by these two properties, so

   $$\lfloor \frac{2011}{k} \rfloor \stackrel{def}{=} |D_k|.$$

   In particular $\lfloor \frac{2011}{15} \rfloor = 134$ and $\lfloor \frac{2011}{120} \rfloor = 16$ taking onto account that $134 < \frac{2011}{15} < 135$ while $16 < \frac{2011}{120} < 17$, hence $|D_{15}| - |D_{120}| = 134 - 16 = 118$.

2. Three boxes are presented to you. One contains $1000, the other two are empty. Each box has a clue written on it as to its contents and only one message is telling the truth, the other two are lying. If the first box says, "The money is not here", the second box says "The money is in the first box" and the third box says, "The money is not here", which box has the money?

   Answer: The third box. Solution: If the money is in the first box, then the messages on both the second and third boxes are true, which contradicts the fact that only one message is telling the truth. If the money is in the second box, then the messages on the first and third box are true, which is again a contradiction. Hence, the money must be in the third box. In that case, the message on box one is true, while the messages on boxes two and three are false as required.

3. It takes a horse and a goat two hours to eat 20 pounds of hay. If it takes the horse three more hours than the goat to eat 20 pounds of hay, how long does it take the horse to eat the 20 pounds of hay?

   Answer: 6 hours. Solution: Let x be the amount of time it takes for the horse to eat the hay. Notice that $x > 3$ since it takes the horse three more hours than the goat to eat 20 lb of hay. Assuming that the horse and the goat are eating at a constant rate, then the horse eats at the rate of $20/x$ lb/hr, while the goat eats at the rate of $20/(x-3)$ lb/hr. We know that if both are eating together they will eat 20 lb of hay in 2 hrs, i.e.,
$$2\left(\frac{20}{x} + \frac{20}{x-3}\right) = 20.$$
   Thus
$$\frac{2x-3}{x^2-3x} = \frac{1}{2}.$$
   Simplifying we find $x^2 - 3x = 4x - 6$, hence $x^2 - 7x + 6 = 0$. So either $x = 1$ or $x = 6$. Since $x > 3$, it must be $x = 6$.

4. The door to the computer room at a school has a keycode. The combination is a sequence of 5 numbers. A student forgot his code. However, he did remember five clues. These are what those clues were:
   (a) The fifth number plus the third number equals fourteen.
   (b) The fourth number is one more than the second number.
   (c) The first number is one less than twice the second number.
   (d) The second number plus the third number equals ten.
   (e) The sum of all five numbers is 30.
   What were the five numbers and in what order?

   Answer: 7, 4, 6, 5, 8. Solution: We can set up a system of equations for x_i =the i-th number in the code, i.e., x_1 is the first number, x_2 is the second number, etc. The system will then be:
$$\begin{array}{cccccl} & & x_3 & & +x_5 & = 14 \\ & -x_2 & & +x_4 & & = 1 \\ -x_1 & +2x_2 & & & & = 1 \\ & x_2 & +x_3 & & & = 10 \\ x_1 & +x_2 & +x_3 & +x_4 & +x_5 & = 30. \end{array}$$
   If we add the third equation to the last one we obtain
$$3x_2 + x_3 + x_4 + x_5 = 31,$$
   while adding together the first, the second and the third equations gives
$$2x_3 + x_4 + x_5 = 25.$$
   If we then subtract the last two equations we obtain an equation for x_2 and x_3
$$3x_2 - x_3 = 6.$$

The latter together with the forth equation of the initial system give

$$\begin{aligned} x_2 \quad &+x_3 \quad = 10 \\ 3x_2 \quad &-x_3 \quad = 6, \end{aligned}$$

which when added show that $4x_2 = 16$. Hence, $x_2 = 4, x_3 = 6$ and after a few substitutions we find that the numbers in order are $7, 4, 6, 5, 8$.

5. Thirty bored students take turns walking down a hall that contains a row of *closed* lockers, numbered 1 to 30. The first student opens all the lockers; the second student closes all the lockers numbered 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30; the third student operates on the lockers numbered 3, 6, 9, 12, 15, 18, 21, 24, 27, 30: if a locker was closed, he opens it, and if a locker was open, he closes it; and so on. For the i^{th} student, he works on the lockers numbered by multiples of i: if a locker was closed, he opens it, and if a locker was open, he closes it. What is the number of lockers that remain open after all the students finish their walks?

   Answer: 5. Solution: Let $d(i)$ be the number of positive integer divisors of the positive integer number i. The i^{th} locker remains open exactly when $d(i)$ is an odd number. This is so because the k^{th} student operates on the i^{th} locker iff k divides i. Now, we use that if $i = p_1^{k_1} p_2^{k_2} \cdots p_j^{k_j}$ is the prime number decomposition of i (i.e., if we write i as the product of powers of distinct prime numbers) we have

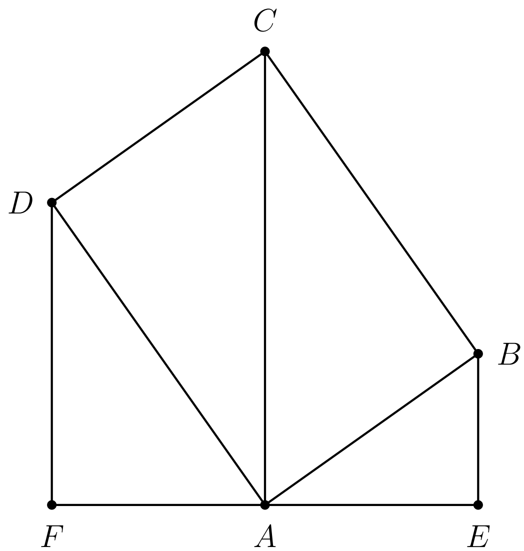
$$d(i) = (1 + k_1)(1 + k_2) \cdots (1 + k_j).$$

   Therefore $d(i)$ is odd if and only if each of the powers $k_1, k_{2,...}, k_j$ is even. This is equivalent to i being a perfect (exact) square. Since there are 5 perfect squares less than 30 the asnwer is 5.

6. A frog makes 2 jumps, each 1 meter in length. The directions of the jumps are chosen independently and at random with equal chances for every direction to be chosen. What is the probability that the frog's final position is at most 1 meter from its starting position?

   Answer: $\dfrac{1}{3}$. Solution: Set up a Cartesian coordinate system for the frog's position. Suppose the frog starts at $(1, 0)$ and his first jump is along the $x-$axis and puts him at $(0, 0)$. Then his second jump must be at angle θ where $-60 \leq \theta \leq 60$ in order for the distance between his final position and his original position to be less than 1. This range of angles is one third of the circle, hence the probability is one third.

7. In the figure below, $ABCD$ is a rectangle. The points A, F, and E lie on a straight line. The segments DF, BE, and CA are all perpendicular to FE. The length of DF is 15 and the length of BE is 6. Find the length of FE.

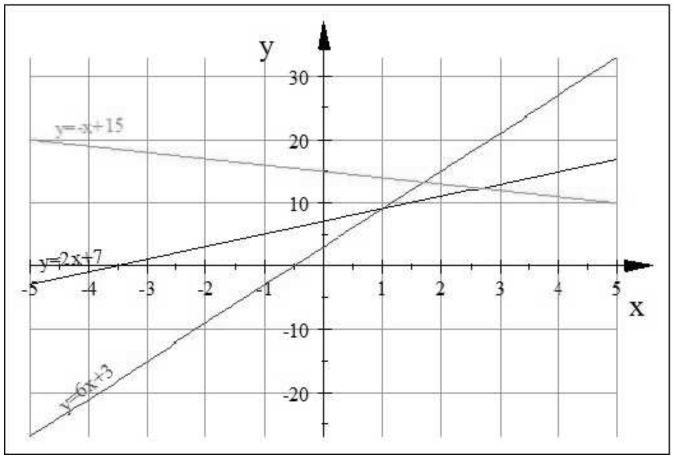


Answer: $6\sqrt{10}$. Solution: Since $ABCD$ is a rectangle its two diagonals bisect each other in equal parts. In particular BD is split in half by AC. Since DF, CA and BE are parallel it follows that CA splits BD and FE in parts which have the same ratios. Therefore A is the midpoint between A and E.

On the other hand, since the angle $\angle DAB = 90^o$ and using again that DF, CA and BE are parallel it follows that the triangles $\triangle DFA$ and $\triangle AEB$ are similar. If we set $AF = x$ from the similarity of the triangles we obtain $x/15 = 6/x$, i.e., $x^2 = 90$ or $x = 3\sqrt{10}$. Thus, $FE = 2x = 6\sqrt{10}$.

8. For each real number x , let $g(x)$ be the minimum value of the numbers $6x + 3$, $2x + 7$, $15 - x$. (For example if $x = 2$ then the three numbers are 15,11,13, so $g(2) = 11$.) Find the maximum value of $g(x)$.

Answer:$\frac{37}{3}$. Solution: Notice that the graph of g is obtained by taking the "lowest" of the graphs of the given functions. In our case, if we graph the lines $y = 6x + 3$, $y = 2x + 7$ and $y = -x + 15$, then the graph of g is the boundary of the region which is below each of the graphs. The graphs of the lines $y = 6x + 3$, $y = 2x + 7$ and $y = -x + 15$ have three points of intersection, which are at $x = 1$, $x = \frac{12}{7}$ and $x = \frac{8}{3}$. For example, the first is obtained by solving $6x + 3 = 2x + 7$, i.e., $4x = 4$ or $x = 1$ (and $y = 9$ of this common point).



Comparing the values of $g(x)$ at the three intersection points, where $g(x)$ is, correspondingly, 9, 93/7 and 37/3, we see that the maximum is 37/3.

9. The triangle $\triangle ABC$ has $AB = 7$ and the given ratio $BC/CA = 24/25$ of the lengths of the other two sides. What is the largest possible area for the $\triangle ABC$?

   Answer: 300 square units. Solution (NM Math Team): Suppose that one has a right triangle, with sides a, b, and c, and with $a^2 + b^2 = c^2$. Now construct another triangle with sides a, bk, and ck, so that two of the sides remain in the same ratio, c/b. Claim: The maximum area of such a triangle is $\frac{1}{2}bc$. Proof: Let the vertices of the constructed triangle be $(-a; 0), (0; 0)$,and $(x; y)$. Taking the ratio of the distances of (x; y) to $(-a; 0)$ and $(0; 0)$ respectively, we have

$$\frac{(x+a)^2 + y^2}{x^2 + y^2} = \frac{(ck)^2}{(ck)^2} = \frac{c^2}{b^2}.$$

   This gives

$$b^2[(x+a)^2 + y^2] = c^2[x^2 + y^2].$$

   Expanding and moving some terms gives $b^2a^2 = (c^2 - b^2)x^2 - 2ab^2x + a^2y^2$, or $b^2a^2 = a^2x^2 - 2ab^2x + a^2y^2$. Dividing out by a^2 gives $b^2 = x^2 - \frac{2b^2}{a}x + y^2$.Now, completing the square for the x terms gives

$$b^2 + \frac{b^4}{a^2} = (x - \frac{b^2}{a})^2 + y^2.$$

   Simplifying with the Pythagorean Theorem again yields

$$\frac{b^2c^2}{a^2} = (x - \frac{b^2}{a})^2 + y^2.$$

   This last is the sum of two squares, and y is clearly maximized when the square term with x is zero, giving

$$y_{max} = \frac{bc}{a}.$$

But this is the altitude of the triangle, giving (maximum) area $A = \frac{1}{2}a \cdot \frac{bc}{a} = \frac{1}{2}bc$. For our special case, $b = 24$ and $c = 25$, giving area 300.

10. Let

$$x = .01234567891011 \cdots 998999$$

where the digits are obtained by listing the numbers 0-999 in order. What is the 2011$^{\text{th}}$ digit to the right of the decimal place?

Answer: 6. Solution: There are 10 digits in the expansion among the one digit numbers and $90 \times 2 = 180$ digits among the two-digit numbers. Hence we are looking for the digit among the three digit numbers which is in the 1821$^{\text{th}}$ place since $2011 - 190 = 1821$. As $3 \times 607 = 1821$ and the 607$^{\text{th}}$ three digit number is $607 + 99 = 706$, the digit in the 2011$^{\text{th}}$ place is 6.