UNM — PNM STATEWIDE MATHEMATICS CONTEST XLIII

February 5, 2011 Second Round Three Hours

1. Find the smallest positive integer n such that every digit of $45n$ is 0 or 4.

Solution: A number is divisible by 9 if the sum of the digits is divisible by 9. As 4 and 9 are relatively prime, if a nonzero natural number whose digits are 4's and 0's is divisible by 9, it must have $9k$ 4's for some $k \geq 1$. So the smallest such number must be 444,444,444. This number however is not divisible by 5. By adding a zero at the end, we obtain $4,444,444,440$ which is divisible by both 9 and 5 and is the smallest such natural number divisible by both. Hence, $n = 98765432$ is the smallest natural number such that $45n$ consists only of digits 4 and 0.

2. At an artisan bakery, French tortes are 52 dollars, almond tarts are 12 dollars and cookies are one dollar each. If Alex has 400 dollars to purchase exactly 100 of these items for a party and he buys at least one of each item, how many of each bakery item does he purchase? Only whole pieces of the bakery items can be purchased.

Solution: Suppose F is the number of French tortes, A is the number of almond tarts purchased and C is the number of cookies purchased. Then we have two equations:

$$\begin{aligned} 52F + 12A + C &= 400 \\ F + A + C &= 100. \end{aligned}$$

Since at least one of each item is purchased then $1 \leq F, A, C$. Since the French tortes are the most expensive, we see that we cannot purchase more than 7 tortes or we will go over the 400 dollars we have to spend. By trial and error we see that we will not obtain integer solutions if $F \neq 2$. However, when $F = 2$, we obtain the two equations in two unknowns:

$$12A + C = 296$$

$$A + C = 98$$

which have solution $A = 18$, $C = 80$.

Alternatively, we solve the system

$$\begin{aligned} 12A + C &= 400 - 52F \\ A + C &= 100 - F \end{aligned}$$

considering F as a parameter. The solution is

$$\begin{aligned} A &= \frac{300}{11} - \frac{51}{11}F \\ C &= \frac{40}{11}F + \frac{800}{11}. \end{aligned}$$

In particular, F is 1, 2, 3 or 4 so that $A > 0$. Only for $F = 2$ we obtain a solution which is of whole numbers.

3. In the following "equation" each letter represents a digit (between 0 and 9). Different letters represent different digits and S is not 0. Determine the digit represented by each of the used letters so that the addition is correct.

$$\begin{array}{r} STORE \\ +\quad STORE \\ STORE \\ \hline TEASE \end{array}$$

Solution: We shall use repeatedly that three times a number between 0 and 9 is a number between 0 and 27.

Adding the units it follows $3 \times E$ equals E, or $10 + E$ or $20 + E$. Solving the three equations we find that either $E = 0$, or $E = 5$, or $E = 10$, which leaves only the first two possibilities.

Now we look at the possibilities for S. Since $3 \times S$ has to be less than 10 it follows that S is either 1, 2 or 3.

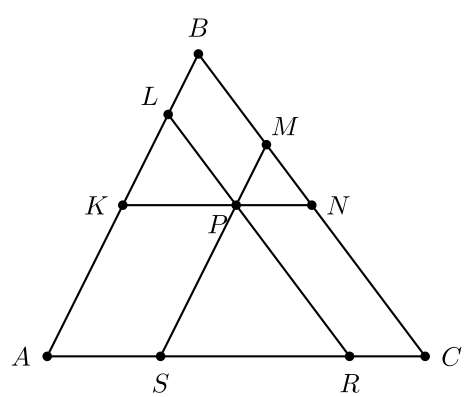
Consider first the case, $S = 1$. The possible values of T are 3, 4 or 5. If $T = 3$ then, taking into account the value of S, it follows that $E = 9$, which is a contradiction. If $T = 4$ then the possible values of E obtained from $3 \times T$ plus any carry over are 2, 3 or 4. This is a contradiction. Finally, if $T = 5$, then E has to be 5, 6 or 7. This is again not possible since different letters represent different digits, which gives a contradiction in the first case, while the last two possibilities are excluded from the already determined values of E.

The case $S = 3$ leads to a contradiction as well since then $T = 9$ which leads to a contradiction since $3 \times T = 3 \times 9 = 27$.

Consider now the case, $S = 2$. The possible values of T are 6, 7 or 8. The first two possibilities are excluded by the possible values of E and the value of S. The last choice gives the solution

$$\begin{array}{r} 28375 \\ +\quad 28375 \\ 28375 \\ \hline 85125 \end{array}$$

4. P is a point inside the triangle $\triangle ABC$. Lines are drawn through P parallel to the sides of the triangle. The areas of the three resulting triangles $\triangle PMN$, $\triangle PLK$ and $\triangle PRS$ are 9, 25 and 81, respectively. What is the area of $\triangle ABC$?



Solution: $\triangle PMN$, $\triangle KLP$ and $\triangle SPR$ are similar triangles. We shall use that if we have two similar triangles with the ratio of linear elements (such as lengths of sides etc.) equal to λ, then the ratio of their areas is λ^2. Let $PN = x$, $PK = y$ and $SR = z$. From the similarity it follows, $y = 5/3x$ and $z = 3x$. Note that $\triangle ABC$ is also similar to $\triangle PMN$ and its base is $x+y+z$, so the similarity coefficient is $(x+y+z)/x = 17/3$. So the areas satisfy the relation $A_{\triangle ABC} = (17/3)^2\ A_{\triangle PMN} = 17^2 = 289$.

5. (a) Let n and d be integers. Show that there are infinitely many integers m such that $n^2 + md$ is an exact square.
   (b) An arithmetic progression is a sequence of numbers of the type

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d, \ldots,$$

where d is the common difference between two successive terms. Prove that if an arithmetic progression of integers contains an exact square (of an integer number), then it contains an infinite number of exact squares.

Solution: (a) $(n+kd)^2 = n^2 + 2nkd + k^2d^2 = n^2 + d(2nk + k^2d)$. Hence, for any integer k, $(n+kd)^2$ is an exact square and has the form $n^2 + md$ with $m = 2nk + k^2d$. As the set of integers is infinite, the set of $m = 2nk + k^2d$ is infinite.

(b) An arithmetic progression is a sequence a_n for n an integer in the form $a_n = r + dn$ for $n \geq 0$ where d is the common difference between terms. If $a_n = r + dn = m^2$ for some n, by (a) there are an infinite number of k such that $m^2 + kd$ is an exact square. Hence a_{n+k} will be an exact square for these k.

6. Find the smallest possible ratio of the radii of two concentric circles centered at a point inside a given triangle so that one of the circles contains while the other is contained in the given triangle. Thus, each of the vertices of the triangle lies inside or on the larger circle, while no point of the smaller circle is outside the triangle.

radius of the smallest circle centered at P containing the triangle (i.e. each of the vertices of the triangle lies inside or on the circle). Let r be the radius of the largest circle centered at P triangle). Find the smallest value R/r can be.

Solution: Let P be the common center of the two circles of respective radii R and r, where R is the radius of the circle containing the triangle and r is the radius of the circle contained in the triangle. We will show that $R/r \geq 2$. Let α_i, $\alpha_i = 1, \ldots, 6$ be the respective angles subtended by a vertex of the triangle and the point P when seen from another vertex of the triangle. It follows

$$180^o = \alpha_1 + \cdots + \alpha_6 \geq 6 \times \alpha,$$

where α denotes the smallest among the angles α_i, $\alpha_i = 1, \ldots, 6$. Clearly, $\alpha \leq 30^o$.

Let A_1, B_1 and C_1 be the ends of the perpendicular segments from P drawn, correspondingly, towards the sides AB, CA and AB of the triangle $\triangle ABC$. We can assume, after possibly renaming the vertices, that $\angle PAC_1 = \alpha$. Consider the right triangle $\triangle PAC_1$ in which we denote $\tilde{r} = PC_1$ and $\tilde{R} = AP$. Since $\angle PAC_1 \leq 30^0$ it follows $\tilde{R} \geq 2\tilde{r}$. On the other hand we have $\tilde{R} \leq R$ and $\tilde{r} \geq r$ since the given circles contain and are contained in the triangle $\triangle ABC$. Therefore

$$R \geq \tilde{R} \geq 2\tilde{r} \geq 2r.$$

i.e., $R \geq 2r$. Notice that equality is obtained for an equilateral triangle.

We can also argue that the ratio R/r is smaller for smaller R and larger r. The smallest circle containing the triangle is the circumscribed circle. the largest circle contained in the triangle is the inscribed circle. Therefore the ratio is as small as possible when the circumscribed and the inscribed circles are concentric. For this to happen the triangle must be equilateral (we leave this as an exercise) in which case we can see that $R/r = 2$.

7. Let z_1, z_2 and z_3 be the roots of the polynomial $Q(x) = x^3 - 9x^2 + 1$. In other words, $Q(z_1) = Q(z_2) = Q(z_3) = 0$. If $P(x) = x^5 - x^2 - x$, what is the value of $P(z_1) + P(z_2) + P(z_3)$?

Solution: Since z_1, z_2 and z_3 are the roots of Q, $z_i^3 = 9z_i^2 - 1$ for each $i = 1, 2, 3$. Thus

$$z_i^5 = 9z_i^4 - z_i^2 = 9z_i(9z_i^2 - 1) - z_i^2 = 9^2 z_i^3 - z_i^2 - 9z_i = 9^2(9z_i^2 - 1) - z_i^2 - 9z_i$$
$$= (9^3 - 1)z_i^2 - 9z_i - 9^2$$

Hence, $P(z_i) = z_i^5 - z_i^2 - z_i = \left((9^3 - 1)z_i^2 - 9z_i - 9^2\right) - z_i^2 - z_i = (9^3 - 2)z_i^2 - 10z_i - 9^2$.

On the other hand, by Vieta's formulas, we have

$$9 = z_1 + z_2 + z_3,\ 0 = z_1z_2 + z_1z_3 + z_2z_3,\ -1 = z_1z_2z_3.$$

Therefore,

$$\begin{aligned} P(z_1) + P(z_2) + P(z_3) &= (9^3 - 2)(z_1^2 + z_2^2 + z_3^2) - 10(z_1 + z_2 + z_3) - 3 \cdot 9^2 \\ &= (9^3 - 2)\left((z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_1z_3 + z_2z_3)\right) - 9 \cdot 10 - 3 \cdot 9^2 \\ &= (9^3 - 2)\left((z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_1z_3 + z_2z_3)\right) - 9 \cdot 10 - 3 \cdot 9^2 \\ &= (9^3 - 2) \cdot 9^2 - 9 \cdot 10 - 3 \cdot 9^2 = 9^2 \cdot (9^3 - 5) - 9 \cdot 10 = 58,554. \end{aligned}$$

8. A very wild ant is moving in a plane with a fixed Cartesian (orthogonal) coordinate system.

a) Suppose the ant is moving parallel either to the x- axis or the y-axis. Suppose the ant travels 3 centimeters per second parallel to the y axis direction (in either direction) and 2 centimeters per second when moving parallel to the x axis (in either direction). If the ant starts at the origin, what region of the plane can the ant reach in one second?

b) Suppose the ant can travel 3 centimeters per second when traveling *on the* y-axis and 2 centimeters per second in any other direction when off the y-axis. If the particle starts at the origin, what region of the plane can the ant reach in one second?

Solution: a) First notice that by traveling only along a coordinate axis the ant can reach in one second the points with coordinates $A(0, 3)$,$B(0, -3)$, $C(2, 0)$ and $D(-2, 0)$. The rhombus formed by these points as vertices (including its interior) is the domain the ant can reach in one minute. Indeed, any point on the boundary can be reached by traveling first along the x-axis and then parallel to the y-axis. For example, the points on the segment AC are described by $t \cdot (2, 0) + (1 - t) \cdot (3, 0) = (2t, 3(1 - t))$ for $0 \leq t \leq 1$. From this representation it follows that the point $(2t, 3(1 - t))$ can be reached by traveling t seconds parallel to the x-axis and $1 - t$ seconds parallel to the y-axis. All points on the other edges of the rhombus can be reached in a similar manner.

b) As stated the problem is a little ambiguous, or even unclear. Namely, two possible interpretations are the following: (1) the ant can travel $3cm/s$ on the y-axis and $2cm/s$ in any direction from a point off the y-axis; (2) the ant can travel from any point at the rate of $3cm/s$ parallel to the y-axis and $2cm/s$ in any other direction. Let us call the first interpretation the "slow" ant and the second the "fast' ant.

The answer is independent of the interpretation even though the motions are quite different. In particular, notice that for the slow ant it matters if the ant goes first along the x-axis and then vertically, or first along the y-axis and then horizontally. In fact, the slow ant will end at a different point when following the sequence of the two directions. For the fast ant the order of a vertical and horizontal motion does not change the end point. Any point that the slow ant can reach can be reached also by the fast ant (but the explanation is not exactly using the meaning of fast and slow!). Conversely, any point that the fast ant can reach can be reached also by the slow ant. This is indeed the case because taking a particular route to some point of the fast ant, we can create a new route by putting all vertical motions at the beginning and then allowing only horizontal travel for the rest of second so that the fast ant ends at the considered point (the order doesn't matter!). This way we created a route, which the slow ant can follow in order to reach the arbitrary point reached by the fast ant. Furthermore, let us observe that taking into account what we already observed the order of any two successive motions of the fast ant does not change the end point.

Because of the above argument it is enough to describe the point the fast ant can reach, and we can even consider only routes where the ant travels along the y-axis only at the beginning of the allowed second and then travels in any direction not parallel to the y-axis.

Consider a curve traced by the ant in one second. Let t be the total time during this second when the ant was traveling along the y-axis. The furthest it can go upwards during this t seconds is $3t$. For the remaining time the ant is moving not-parallel to the y-axis. The speed of the ant during this remaining $1-t$ seconds is 2. Therefore the ant can reach any of the points (x, y) which are on or inside the circle of radius $2(1-t)$ from the point $(0, 3t)$ on the y-axis. The argument we just did can be done also for "downward' travels, which gives the region of the plane the ant can reach in one second:

$$\cup_{0\leq t\leq 1}\left\{(x,y) \mid x^2+(y-3t)^2\leq 4(1-t)^2\right\}\frac{x^2}{4}+\frac{y^2}{9}\leq 1\}.$$

What is the exact envelope of these circles requires a further argument that was not expected on this test. It looks like a circle with a roof and its symmetric image w.r.t. the x-axis attached. You can visualize the envelope by looking at the following animation and then you can read about envelopes of curves in order to learn how to find them.

9. The faces of a solid figure are all triangles. The figure has 11 vertices. At each of six vertices, four faces meet and at each of the other five vertices, six faces meet. How many faces does the figure have?

Solution: Each vertex is counted as part of three faces, so the number of faces f equals $f = (6\times 4+5\times 6)/3 = 54/3 = 18$.

Alternatively, we can use Euler's formula $v - e + f = 2$ where v is the number of vertices, e is the number of edges and f is the q number of faces, see December 2010 problems of the month http://www.unm.edu/˜vassilev/pbs.html. Here $v = 11$. We need to first determine e then we can find f. If four faces meet at a vertex this means that there are four edges at this vertex. Similarly if six faces meet at a vertex, there are six edges at that vertex. Since each edge starts and ends at different vertex, then $2e = 6\times 4+5\times 6 = 54$. Hence $e = 27$ and $f = e - v + 2 = 27 - 11 + 2 = 18$.

10. Nine scientists are working on a secret project. They wish to lock up the documents in a cabinet so that the cabinet can be opened when and only when *five or more of the scientists are present*. For this purpose a certain number of locks are installed on the cabinet and each of the scientists is given keys to some of these locks. Each key can open exactly one lock. Thus, for the cabinet to be opened (i) any five of the scientists have to be present and (ii) the keys to all of the locks on the cabinet have to be among the set of all keys given to the present five scientists. What is the smallest number of locks needed?

Solution: In the first step we will find a number of locks so that no less locks can solve the problem. Since no set of four scientists can open the cabinet for each combination of five scientists there must be at least one lock that cannot be opened by this group of four people. In this way to every combination of four scientists we associate a set of "missing" keys. Furthermore, since every five scientists can open the cabinet the "missing" keys of any two different combinations of four scientists have to be two disjoint sets. For otherwise there will be two groups of four scientists missing the same key, so there will be five scientists missing a key, which is a contradiction. There are $\binom{9}{4} = 126$ combinations of four scientists, so there must be at least 126 locks on the cabinet.

Next, we turn to the second step, where we shall show that 126 locks do solve the problem. For this we have to show that there is a way to distribute keys from 126 locks in a way which satisfies conditions (i) and (ii). It will be essential that

$$\binom{9}{4} = \binom{9}{5}, \tag{*}$$

which is true since selecting any five is the same as designating the remaining four. So, lets make five keys for each of the 126 locks. Then for each of the 126 combinations of five scientists (here we use (*)) give a key to the same lock to each of the scientists in the combination. In this way, each group of five scientists has a lock which they and only they can open. In particular, the remaining four cannot open it, so no less than five scientists can open the cabinet. We still have to show that every five scientists can open the cabinet, i.e., each group of five scientists has 126 different keys among them. For this we note that given any lock there is a distinguished combination of five scientists who all have keys to it. On the other hand, if we take any five scientists, at least one of them will belong to the distinguished combination since there are nine scientists overall. This proves the final claim.

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