UNM - PNM STATEWIDE MATHEMATICS CONTEST XLIII

November 5-8, 2010 First Round Three Hours

1) For how many integers x in the set $\{1, 2, \ldots, 2010\}$ is $x^4 - x^3$ a cube?

Answer: 13.

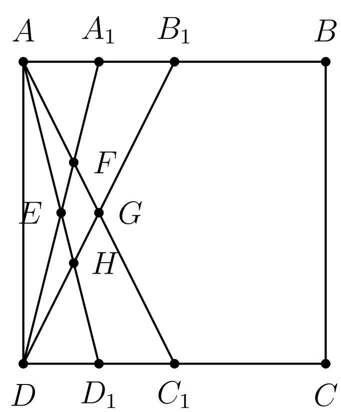
Solution: $x - 1$ has to be a cube, so the solution is given by the number of integers z such that $x = z^3 + 1 \leq 2009$. So $z^3 \leq 2009$. Since $12^3 < 2009$ and $13^3 > 2009$ it follows there are thirteen z's, $z = 0, 1, \ldots, 12$, hence, 13 x'x as required.

2) In how many ways can 5 different prizes be awarded to 4 students so that each student receives at least one prize?

Answer: 240.

Solution: One of the students must receive two prizes while the rest will receive one prize each. Thus, the number of different distributions of the awards is the same as the number of ways we can choose two of the prizes, which can be done in $\binom{5}{2} = \frac{5 \cdot 4}{2} = 10$ ways, and then order the remaining four. The number of possible ways we can order for elements is $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ (four choices for the first place, three for second, etc.), and thus there are $\binom{5}{2}\ 4! = 240$ ways to distribute the awards in the required manner.

3) If $ABCD$ is a square with side AB of length 1cm and the segments AA_1, A_1B_1, C_1D_1 and D_1D all have length $\dfrac{1}{4}$cm, what is the area of the quadrilateral $EFGH$? See the figure for the location of the points.



Answer: 1/48.

Solution: Using the similar triangles $\triangle AFA_1 \sim \triangle C_1DF$, the first one is twice smaller than the second, we see that F is at a distance $2/3$ from DC. Similarly we see that H is at a distance $1/4$ from DC. On the other hand, both, E and

G are at a distance $1/2$ from DC. Now using the areas of the triangles $\triangle DC_1F$, $\triangle DD_1E$, $\triangle DC_1G$ and $\triangle DD_1H$ we see that the area of the quadrilateral $EFGH$ is

$$Area_{\triangle DC_1F} - Area_{\triangle DD_1E} - Area_{\triangle DC_1G} + Area_{\triangle DD_1H}$$
$$= \frac{1}{2}\left(\frac{2}{3}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{4}\right) = \frac{1}{48}.$$

4) Let x and y be real numbers satisfying the equations

$$x + y = 1, \qquad (x^2 + y^2)(x^3 + y^3) = 26.$$

What is $x^2 + y^2$?

Answer: $13/3$.

Solution: We use $x^2+y^2 = (x+y)^2 - 2xy = 1 - 2xy$. Next we determine xy. Let $p = xy$. From $26 = (x^2+y^2)(x^3+y^3) = ((x+y)^2 - 2xy)\,((x+y)^3 - 3xy(x+y)) = (1-2p)(1-3p) = 6p^2 - 5p + 1$ we see that $6p^2 - 5p - 25 = 0$. Since $6p^2 - 5p - 25 = (3p+5)(2p-5)$ solving for p we see that $p = -5/3$ or $p = 5/2$. Thus, $x^2+y^2 = 13/3$ since $x^2 + y^2 \geq 0$ when x and y are real numbers.

5) How many triples of real numbers (a, b, c) satisfy the equations

$$ab = c, \quad bc = a, \quad ca = b?$$

In other words, how many triples of real numbers (a, b, c) have the property that the product of any two equals the third of the numbers?

Answer: 5.

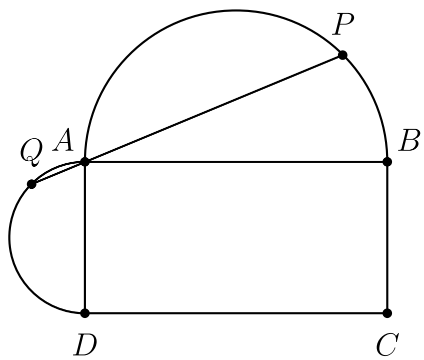
Solution: If one of the numbers is zero, then clearly all of them must be zero. Suppose none of the numbers is zero. Multiplying all the equations we see that $a^2b^2c^2 = abc$ and thus $abc = 1$. Since $ab = c$ it follows $c^2 = 1$, i.e., $c = \pm 1$. If $c = 1$ the other two of the given equations imply that either $a = b = 1$ or $a = b = -1$. If $c = -1$ the other two of the given equations imply that either $a = 1 = -b$, or $-a = 1 = b$. Thus the solution is given by the triples $\{(0,0,0), (1,1,1), (-1,-1,1), (1,-1,-1), (-1,1,-1)\}$.

6) A palindrome is a number which reads the same backwards and forwards, for example 272 and 1331 are a palindromes. Diego decided to use a calculator to add the numbers from 1 to 15. Upon doing this he got a palindrome. Diego was certain that he entered every number correctly, except he forgot to add one of the numbers. What was the number he missed?

Answer: 9.

Solution: The sum of the whole numbers from 1 to 15 is $\frac{15\cdot 16}{2} = 120$. The nearest palindrome smaller than 120 is 111, so Diego missed 9 in his calculation.

7) Semicircles are drawn on two sides of a rectangle $ABCD$ in which the longer side AB is twice the length of the shorter side AD, as shown. QAP is a line segment with segment QA of length $5cm$ and segment AP with length $24cm$. Compute the length of the shorter side.



Answer: 13

Solution: We give a solution using similar triangles and the Pythagorian theorem.

First, we note that $\triangle DAQ$ is similar to $\triangle ABP$, symbolically, $\triangle DAQ \sim \triangle ABP$. Indeed, both triangles have one angle equal to 90^o while at the vertex A we have $\angle QAD + \angle PAQ = 90^o$ since $\angle DAB = 90^o$. Since the sum of all angles in a triangle is 180^o it follows that $\angle QAD = \angle PBA$.

Now that we know that $\triangle DAQ$ and $\triangle ABP$ are similar, we note that $\triangle ABP$ is twice larger than $\triangle DAQ$ since we are given that the side AB of the rectangle is twice longer than the side DA. Hence

$$\frac{DA}{AB} = \frac{AQ}{BP} = \frac{DQ}{AP}.$$

Let $DA = x$ and $AB = 2DA = 2x$ be the respective lengths. The above identity can be written as

$$\frac{x}{2x} = \frac{5}{BP} = \frac{DQ}{24},$$

which shows that $DQ = 12$. Now, the Pythagorian theorem applied to $\triangle DAQ$ gives $x^2 = 12^2 + 5^2$, i.e., $x = 13$.

8) Find the radius of the largest disc that can be covered by three discs each of radius R.

Answer: The radius of the circle circumscribed around the equilateral triangle with sides $2R$ or $\frac{2\sqrt{3}}{3}R$.

Solution: That the disc with this radius can be covered is clear by placing the centers of the given discs at the mid-points of the sides of the equilateral triangle. A larger disc cannot be covered. Indeed, since every point on the boundary circle of this disc has to be covered and each one of the given three discs covers an arc of this circle it follows that there will be an arc of length at least 1/3 of the total length covered by one of the given three discs. Thus there will be a chord of length strictly greater than $2R$ that is also covered by this given disc, which is a contradiction.

9) Two cities A and B are 365 miles apart. A blue marker is placed every 36/365 miles from A on the road to B. A red marker is placed every 14/365 miles from A on the road to B. What is the shortest distance between markers of any color?

Answer: 0.

Solution: We are looking for the minimum of $\left|\frac{36}{365}x - \frac{14}{365}y\right| = \frac{2}{365}\left|18x - 7y\right| = 0$ by taking $x = 7$ and $y = 18$ (the least common multiple $lcm(18, 7) = 18 \cdot 7$). Since $36 \cdot 7 = 252$ and $14 \cdot 18 = 252$ and both numbers are less than 365 there will be a blue and a red marker at a point between the two cities. Thus, the shortest distance between markers of any color is 0.

10) A school has a number of club activities that are practiced weekly during the school year which has thirty weeks. Before the start of the school year every student has to select at least one but no more than twelve different club activities. After the selection the student has to put the selected clubs in order which they follow cyclicly throughout the year attending each week the corresponding club from the chosen order. However the number of selected clubs of each student has to be such that the cycle is completed during the last week of classes.The following data was recorded during the school year for five chosen girls.

(a) [(i)]

(b) The five chosen girls Bea, Donna, Ellen, Ginny and Izzy selected different number of clubs.

(c) During the first week, Bea attended track, Donna and Ellen attended dance, Ginny attended mathematics and Izzy attended science.

(d) During the eleventh week, two attended dance, one attended mathematics, one attended track and one attended history.

(e) During the nineteenth week, Ginny attended mathematics and Izzy science, and the remaining girls went to dance.

(f) Ginny attended science during the twenty-second and history during the twenty-third week.

Who among these five girls attended the math club during the eleventh week?

Answer: Izzy.

Solution: We are given that each of the five students has selected certain number of different clubs, which are then put in order which they follow cyclicly for the entire year attending one club every week (since every girl was "attending each week the corresponding club from the chosen order"). Since the school year has 30 weeks and every cycle of selected clubs has to be completed during the last school week the number of clubs each of the five girls selected has to be one of the numbers 1, 2, 3, 5, 6, 10 or 15, which are the divisors of 30. We will also call these numbers "cycles" of each girl. Furthermore, since every student can select at most 12 clubs and among these five girls every two selected different number of clubs the set of possible cycles of the five girls is reduced to 1, 2, 3, 5, 6, or 10. This information can be summarized in the following table.

	01	11	19	22	23	length of cycle
B	t		d			
D	d		d			
E	d		d			
G	m		m	s	h	
I	s		s			
possible		d,d,m,t,h				1,2,3,5,6 or10

A key observation that follows from the fact that all chosen clubs in a cycle are different is that if during any two weeks a girl is taking the same club then these two weeks must have the same remainder when divided by the length of the cycle. So, for example, if dance is taken during the 1st week and the cycle has length 10 then dance will be taken again in the 11th, 21st etc. weeks and only then, i.e., only in the weeks that have remainder 1 when divided by 10. Armed with this key observation we continue to analyze the problem.

From the given data it follows that 5 or 10 cannot be the length of the cycle of any of the last four girls, Donna, Ellen, Ginny and Izzy, since in the 19th week each one of them had the same club as in the 1st week. Thus, the last four girls have cycles of lengths (all different!) among 1, 2, 3 and 6. Notice how 1 and 19 have the same remainder when divided by any of these numbers.

The above implies immediately that the first girl, Bea, has a cycle of length 5 or 10. In particular in the 11th week she must have had the same club as in the first week, so she is the girl who had track in the 11th week.

Using the data (v) for Ginny we see that the length of her cycle cannot be 1, 2 or 3 since $22 \equiv 19 \mod 3$ and $\mod 2$. Thus, the length of her cycle is 6. In particular, she did not take the math club during the 11th week, hence she had dance or history that week.

At this point we have that Donna, Ellen and Izzy have cycles of lengths in the set 1, 2 , 3, and the clubs are dance, math, history or dance, dance, history. Since 1 and 11 have the same remainder when divided by 1 or 2, but not when divided by 3, Izzy must have had a cycle of length 2 or 3. However, it cannot be 2 since "science" is not in the possible clubs in week 11 and Izzy had "science" the 1st week. Thus, Izzy's cycle is of length 3 and she attended the math club during the 11th week.

	01	11	19	22	23	cycle length
B	t	t	d			5 or 10
D	d		d			
E	d		d			
G	m		m	s	h	6
I	s	m	s			3
possible		d,d,m,t,h				1,2,3,5,6 or10

Remark: Since we already did so much work, let's see how much exact information can we determine. At this stage we have the data in the above table. Since Ginny's cycle is of length 6 and she had history in fifth place (by the information in weeks 19 and 23) it follows that Ginny had history in week 11. this leaves Donna and Ellen with math in their 11th week and one of them has chosen only one club while the other has chosen two clubs.

	01	11	19	22	23	cycle length
B	t	t	d			5 or 10
D	d	d	d			
E	d	d	d			
G	m	h	m	s	h	6
I	s	m	s			3
possible		d,d,m,t,h				1,2,3,5,6 or10

Following are solutions submitted to us by Bill Cordwell on behalf of the NM Math Team