UNM - PNM STATEWIDE MATHEMATICS CONTEST XLI

February 7, 2009 Second Round Three Hours

(1) An equilateral triangle is inscribed in a circle which is circumscribed by a square. This square is inscribed in a circle which is circumscribed by a regular hexagon. If the area of the equilateral triangle is 1, what is the area of the hexagon?

Proof. Note, that $1 = \frac{3\sqrt{3}}{4} r_1^2$ and $r_2 = \sqrt{2} r_1$. The area of the hexagon is $\frac{6r_2^2}{\sqrt{3}} = \frac{12r_1^2}{\sqrt{3}} = \frac{48}{9} = \frac{16}{3}$. □

(2) Suppose you have a piece of paper which you cut into either four or sixteen pieces. After that you cut again some of the pieces into either four or sixteen smaller pieces. Suppose you have nothing else to do, so you keep repeating this procedure cutting some of the pieces into either four or sixteen smaller pieces. Can you end up with 2009 pieces at some stage of your cutting process?

Proof. Every time we increase the number of pieces by $3k + 15l$, where k is the number of pieces that we cut into four, and l is the number of pieces we cut into sixteen. Since we start with one piece of paper, in order to have 2009 pieces we must have $2009 = 1+3n+15m$ for some natural numbers n and m. Thus $2008 = 3n+15m$, which is impossible since 3 does not divide 2008. □

(3) A group of 200 high school students visited the University of New Mexico for Lobo Day. The students could participate in at most two of the following three workshops: 1) Algorithms, 2) Bioinformatics and 3) Coding. Suppose 100 students participated in Algorithms, 90 participated in Bioinformatic, 80 participated in Coding.
  (a) If 87 participated in both Algorithms and Bioinformatics or both Algorithms and Coding. How many students participated in both Bioinformatics and Coding?
  (b) If in addition 24 participated in Coding but not in Algorithms or Bioinformatics, how many participated in both Algorithms and Bioinformatics?

Proof. (a) Let A, B and C denote the sets of students participating, correspondingly, in the Algorithms, Bioinformatics and Coding workshops. Since the students could participate in at most two workshops we have that the intersection $A \cap B \cap C = \emptyset$. Using the Inclusion-Exclusion Principle, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$, we see that $|A \cup B \cup C| + |B \cap C| = (100 + 90 + 80) - 87 = 183$ since $|A \cap B \cap C| = 0$ and $|A \cap B| + |A \cap C| = 87$. This shows that at most 183 students participated in some workshop (even though 200 visited the University), and 13 students participated only in Algorithms. Let us define the *non-negative integer variables* $x = |A \cap B|$, $y = |A \cap C|$ and $z = |B \cap C|$. For example, in the case when exactly 183 students participated in some workshop we have $z = 0$ (the minimum possible value), while x and y could be any solution of $x + y = 87$, see Fig. 1. We could have less than 183 participants though, in which case z

will be positive. For example we could have 181 participants as in the situation depicted in Fig. 2.

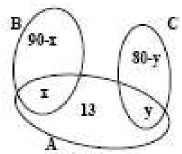


Fig. 1

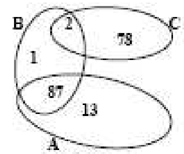


Fig. 2

However, we cannot have too few participants since each student can participate in at most two workshops. What is then the minimum number of participants? Notice that the smaller $|A \cup B \cup C|$ is, the bigger z must be (since the sum of these two number is always 183). The question then is to find what is the maximum possible value of z. Since $x = 87 - y$, we reduce to finding the biggest z, such that, for some y

$$z \geq 0, \qquad y \geq 0, \qquad z + y \leq 80 \qquad \text{and} \qquad 87 - y + z \leq 90,$$

recall that we are dealing with non-negative integer variables. Geometrically, we have to find among all points with integer coordinates in the first quadrant and below the lines $y + z = 80$ and $z - y = 3$ the point(s) which have the biggest z coordinate, i.e., the "highest" among all such points. Drawing the graphs of the lines $y + z = 80$ and $z - y = 3$ we see that the maximum possible value of z is 41, which is achieved for $y = 38$ or $y = 39$. On Fig. 3 you can see the graphs , while Fig. 4 shows a zoomed picture near the point of intersection of the two lines.

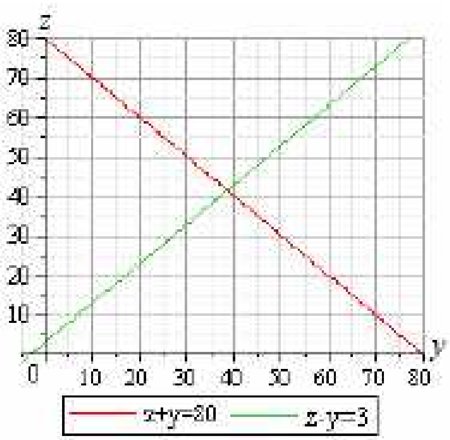


Fig. 3

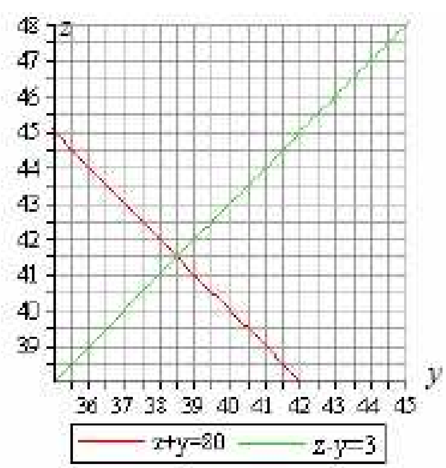


Fig. 4

Thus the maximum possible value of z is 41, see Fig. 5 and Fig. 6.

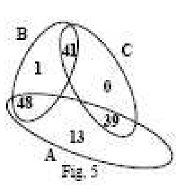


Fig. 5

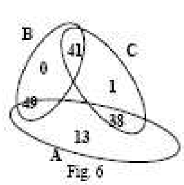


Fig. 6

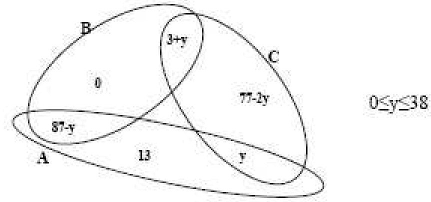


Fig. 7

Furthermore, $|B \cap C|$ could have any number of students between 0 and 41, for example, one solution is shown in Fig. 7.
To sum up, under the given conditions we must have at least 142 but no more that 183 students participating, and any any number between these two extremes is possible. Correspondingly, the number of students taking Bio-informatics and Coding can be any number between zero and forty one.

(b) With the notation from part a) it follows that $y = 56 - z$, hence $x = 87 - (56 - z) = 31 + z$, where $0 \le z \le 41$, see Fig. 8.

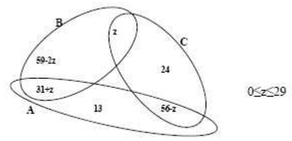


Fig. 8

However at this point we compute $|B \setminus (A \cup C)| = 90 - (31 + z) - z = 59 - 2z$, i.e., $59 - 2z$ students participated in Bio-informatics only.Thus z can be at most $59/2$, i.e., 29 since z is an integer. Therefore, since $x = 31 + z$ we have $0 \le x \le 60$. In conclusion, the number of students taking Algorithms and Bioinformatics can be any number between zero and sixty.

□

(4) Find the smallest N such that $\frac{1}{3}N$ is a perfect cube, $\frac{1}{7}N$ is a perfect seventh power and $\frac{1}{8}N$ is a perfect eighth power.

Proof. Since N is a perfect eighth power, N must be non-negative. Thus $N = 0$ is the smallest number having the required properties.

However, we might want to know also what is the smallest positive integer number N which satisfies the given conditions. In this case, there are positive integers a, b, c such that $3a^3 = 7b^7 = 8c^8$. These equalities force c to be divisible by 21, b to be divisible by 24 and a to be divisible by 14. Hence, N must be divisible by 3^{56}, 7^{24} and 2^{21}, i.e. $N = 3^{56k}7^{24m}2^{21n}$. Also $56k - 1$ must be divisible by 3, $24m - 1$ must be divisible by 7 and $21n - 3$ must be divisible by 8. The smallest $k = 2$, the smallest $m = 5$ and the smallest $n = 7$. Hence, $N = 3^{112}7^{120}2^{147}$.

□

(5) Determine the right triangle of smallest perimeter with integer sides, which has the property that the area is equal to three times the perimeter.

Proof. By the Pythagorean Theorem, we know that $a^2 + b^2 = c^2$, where a and b are the perpendicular sides and c is the hypotenuse. The perimeter of the right triangle is $a + b + c$ and the area is $\frac{ab}{2}$. We want to find all integers a, b, c satisfying $\frac{ab}{2} = 3(a + b + c) = 3(a + b + \sqrt{a^2 + b^2})$. We obtain $36(a^2 + b^2) = 36(a^2 + 2ab + b^2) - 12(a^2b + ab^2) + a^2b^2$ or $ab(72 - 12a - 12b + ab) = 0$. Since, $a, b > 0$ then $72 + 12a + 12b + ab = 0$. This is equivalent to $(a - 12)(b - 12) = 72$. The factor pairs of 72 are $(\pm 1, \pm 72)$, $(\pm 2, \pm 36)$, $(\pm 3, \pm 24)$, $(\pm 4, \pm 18)$, $(\pm 6, \pm 12)$, $(\pm 8, \pm 9)$. Except for the pair $(-8, -9)$, the other negative pairs each force a side of the triangle to have non-positive length. Also, the pair $(-8, -9)$ would yield a 3,4,5 right triangle which clearly doesn't satisfy $\frac{3 \cdot 4}{2} \neq 3(3 + 4 + 5)$. Hence, the possible solutions are $(a, b) \in \{(13, 84), (14, 48), (15, 36), (16, 30), (18, 24), (20, 21)\}$. The triangles will have sides $(13, 84, 85), (14, 48, 50), (15, 36, 39), (16, 30, 34), (18, 24, 30), (20, 21, 29)$. □

(6) 2009 lines are drawn in the plane such that:
- No two lines are parallel.
- At all points of intersection, at least three lines meet.

Show that all the lines go through one point.

Proof. Under the given conditions, there are finitely many points through which at least three of the given lines are passing (call them points of intersection). Suppose the claim is not true. Therefore we can find a point O contained in at least three of the given lines, at a positive distance from some other line l, and such that all other points of intersection are at a bigger or equal distance to the closest line. Since there are at least three lines through O, consider their intersections with the line l (no two lines are parallel!) The "middle" of these points on l will be closer to one of the lines through O, than O is to l.

An alternative solution is to use an induction argument and show that the claim is true for any finite number of lines with the given property. □

(7) The hat game is a collaborative game played by a team of 3 players. Either a red or a blue hat is put on each player's head. The players can see the other players hat colors, but cannot see and do not know the color of the hat on their heads. Each player must either guess the hat color on his/her head or pass. The team wins if every player who does not pass guesses the correct hat color and at least one player does not pass (i.e. makes a guess). Before playing the team can determine a strategy. Is there a strategy to win more than 50 percent of the time? If so, what is the strategy and what is the probability that the team will win?

Proof.
(1st solution) Yes, there is a strategy to win more than 50 percent of the time. Note that the possible combinations of hat placements are

$$RRR, RRB, RBR, BRR, RBB, BRB, BBR, BBB$$

where R =red and B =blue. Three fourths of the time, there are at least two hat colors. If the group decides that the strategy is to only guess blue when a player sees two reds or red when a player sees two blues, then the group fails only when the colors are BBB or RRR. Thus the group will win three fourths of the time. The same result is achieved by only guessing red when a player sees two reds or blue when a player sees two blues, though, it is important that all people use the same caller when they see two hats of the same color.

(2nd solution by Todor Parushev, United World College) The group can do even better, if they can decide on an order in which they guess or pass. For example, suppose the strategy is as follows. (i) The first and second players pass when the third player has a blue hat, in which case the third player guesses blue. Thus, the team wins if the third player has a blue hat. (ii) If a player guesses the players after this player pass. (iii) The first player passes if the third player has a red hat and the second player has a blue hat, in which case the second player guesses blue. Thus, using (ii) and (iii), the team wins when the hats are, in order of the players, BBR or RBR. (iv) The first player guesses red or blue when both the second and third players have red hats. Thus, using (ii) and (iv), the team wins half of the time when the hats are, in order of the players, BBR or RBR.

With the above strategy the team wins in seven out of the eight possible cases, i.e., with probability 7/8.

□

(8) Suppose the beam of each of four projectors lights an oval-shaped area on the stage of a theater. Show that if the spotlights of every three of the given four projectors overlap somewhere, then there is a place which is in the spotlight of all of the projectors.

Proof. The key here is that an oval is a convex figure, i.e., given any two points in the interior of an oval the joining segment lies inside the oval.

Let A_{123} be a point in the spot-light of the 1st, 2nd and 3rd projectors. Define three more points A_{124}, A_{134} and A_{234} in the corresponding spotlights of three of the given four projectors. If the quadrilateral with vertices at these four points is convex, then the intersection of the two diagonals is a point in the intersection of the four ovals - all points on a diagonal lie in the spot-lights corresponding to the common indices of the two vertices. If the quadrilateral with vertices $A_{123}, A_{124}, A_{134}$ and A_{234} is concave, then one of the points lies in the interior or on the triangle formed by the remaining three points, which shows that the "interior" point is in the spot-light of all of the projectors. □

(9) Let a be a non-negative number, i.e., $a \geq 0$. Define successively an infinite sequence of non-negative numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ by letting $a_1 = a$ and then using the formula

$$a_{n+1} = \frac{1}{2}(a_n^2 + 1)$$

for $n = 2, 3, \ldots$ (n runs through all positive integer numbers).

(a) Show that if $0 \leq a < 1$ then all of the numbers in the sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

are less than one.

(b) Show that if $a > 1$ then the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ has arbitrarily large numbers, i.e., given any number M there is at least one number among $a_1, a_2, a_3, \ldots, a_n, \ldots$ which is larger than M.

Proof. If b is a number, $0 \leq b < 1$, then $b^2 < 1$ hence $\frac{1}{2}(b^2+1) < 1$. Part (a) follows.

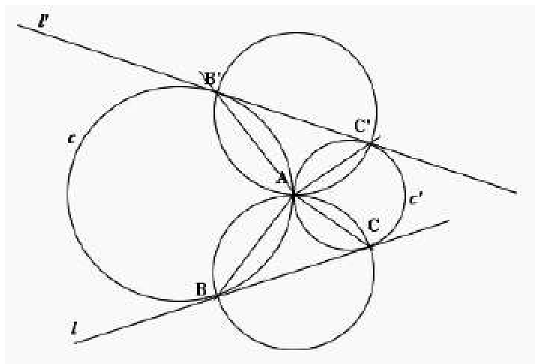
Suppose $a > 1 + x$ for some $x > 0$. Then $\frac{1}{2}(a^2+1) \geq 1 + x + \frac{x^2}{2} > 1 + x$ and $\frac{1}{2}(a^2+1) > a + \frac{x^2}{2}$. Repeating this argument (or using an induction argument) we see that

$$a_{n+1} > a_n + \frac{x^2}{2} \qquad \text{and} \qquad a_{n+1} > 1 + x$$

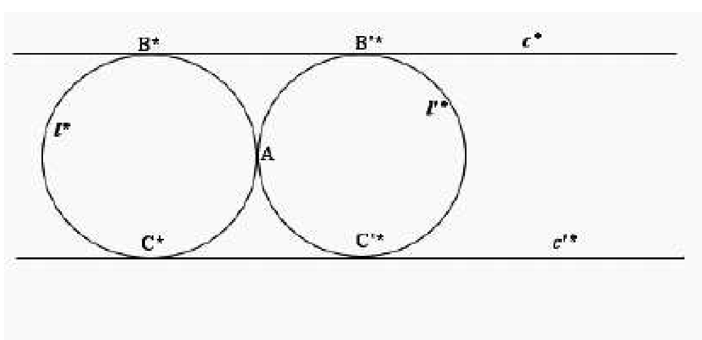
for all positive integers n. Thus $a_n > a_1 + (n-1)\frac{x^2}{2}$, which can be as large as we want.

An alternative way to solve the problem is to use that the sequence $a_1, a_2, \ldots, a_n, \ldots$ is increasing, so if it is bounded it will approach some number A. By continuity, the relation $a_{n+1} = \frac{1}{2}(a_n^2+1)$ will imply that $A = \frac{1}{2}(A^2+1)$, i.e., $A = 1$. This is a contradiction since the first element is bigger than 1. □

(10) Let A be the only common point of two disks, not necessarily of the same radius, with boundaries the circles c and c'. Suppose we draw two lines l and l' which are tangent to each of the circles. Let B and C be the points of contact of the line l, correspondingly, with c and c'. Similarly, let B' and C' be the points of contact of the line l', correspondingly, with c and c'. Show that the circles circumscribed around the triangles $\triangle ABC$ and $\triangle AB'C'$ are tangent to each other.



Proof. Let s and s' be the circles we are interested in. Under an inversion with center the point A, the lines l and l' become congruent circles through A, while the circles c and c' are transformed into two parallel lines touching the images of l and l'. Therefore the images of s and s' are parallel, which shows that the pre-images have only one point in common, namely the point A.



(2nd proof) Let O and O' be, correspondingly, the centers of the circles c and c', see Fig. 1. The two circles are tangent to each other at A and thus OA and $O'A$ are perpendicular to the common tangent line. This implies that the point A lies on the segment OO'. The center K of the circle s' circumscribing the triangle $\triangle B'AC'$ is at equal distances from the three vertices and thus lies at the intersection of the perpendicular bisectors of the segments $B'A$ and AC', see Fig. 2. In fact, K is the midpoint of the segment $B'C'$ (so Fig. 2 is not exact but we shall improve on it!). In order to prove this fact, notice that the triangles $\triangle OAB'$, $\triangle O'AC'$, $\triangle KAB'$ and

$\triangle KAC'$ are isosceles as two of their sides are radii of the same circle. Since the sum of the angles at the point A is 180^o, by considering the just mentioned triangles and using that the angles across equal sides in a triangle are equal, we conclude that the sum of the angles at B' and C' is 180^o. Therefore, the point K lies on the segment $B'C'$, see Fig. 3, and KA is perpendicular to OO'. The latter means that the circle s' is tangent to OO'. By symmetry, or relabeling the points, the circle s is also tangent to OO', hence s and s' are tangent to each other at the point A.

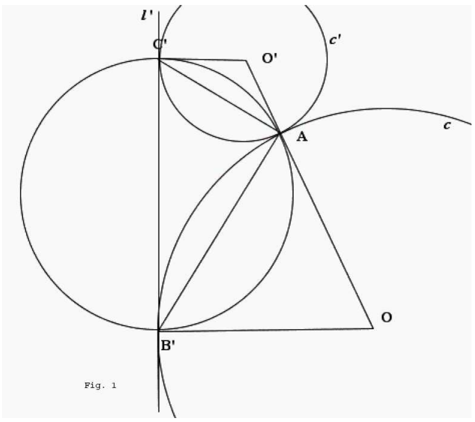


Fig. 1

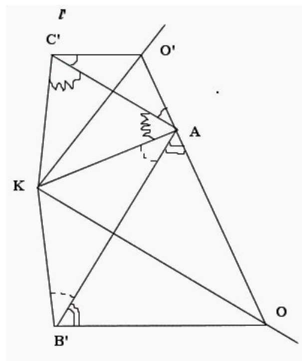


Fig. 2

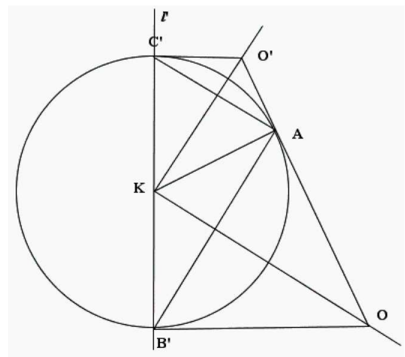


Fig. 3

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