UNM - PNM STATEWIDE MATHEMATICS CONTEST XLI

SOLUTIONS

November 8, 2008 First Round Three Hours

1) Our number system is base 10. Every number can be written as a sum of a digit from 0 to 9 times a power of 10, i.e. $239 = 2 \cdot 10^2 + 3 \cdot 10^1 + 9 \cdot 10^0$. In base n, a number is written as a number in the set $\{0, 1, \ldots, n-1\}$ times a power of n. How would you write our current year, 2008, in the base 2?

Solution: Since $1 \cdot 2^{10} + 1 \cdot 2^9 + 1 \cdot 2^8 + 1 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^4 + 1 \cdot 2^3 = 2008$, then 2008 in base 2 is 11,111,011,000.

2) How can you assign the numbers 1, 3, 5, 7 and 9 to c, o, k, i, e so that the numerical equivalent to cookie is divisible by 3?

Solution: A number is divisible by 3 if the sum of the digits is divisible by 3 †. We conclude that $c + 2o + k + i + e$ must be divisible by 3. Note that $1 + 3 + 5 + 7 + 9 = 25$. If you add an additional 1,3,7 or 9 the number you achieve is not divisible by 3, but if you add an additional 5, you get 30 which is divisible by 3. Thus, 5 must be o and there are 24 choices for assigning c,k,i,e to the numbers in the set $\{1, 3, 7, 9\}$. Overall, we have 24 correct answers - the only restriction is that we assign 5 to o.

† This can be seen, for example, as follows. Note that if a and n are natural numbers, $n > 1$, then $10^n a - a = a(10-1)(10^{n-1} + 10^{n-2} + \cdots + 1) = 9a(10^{n-1} + 10^{n-2} + \cdots + 1)$, which implies that $10^n a$ is divisible by 3 iff a is divisible 3. In other words $10^n a$ and a have the same remainder when divided by 3. More generally, let N be a natural number, $N = 10^n a_n + 10^{n-1} a_{n-1} + \cdots + 10 a_1 + a_0$, where the numbers a_i belong to the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} (in other words $a_n a_{n-1} \ldots a_1 a_0$ is the decimal form of N and the a_i are its digits). A small calculation shows that $N - (a_n + a_{n-1} + \cdots + a_0) = 9a_n(10^{n-1} + 10^{n-2} + \cdots + 1) + \cdots + 9a_1$. Notice that the right hand side of the last equality is divisible by 3 since 9 is divisible by 3. This implies that N is divisible by 3 iff $a_n + a_{n-1} + \cdots + a_0$ is divisible by 3, which is what we wanted to prove. Can you prove other divisibilty criteria that you might know (or find out by doing this exercise)?

3) Equilateral triangles are inscribed in and circumscribed outside a circle. If the area of the larger triangle is 15 square units, what is the area of the smaller triangle? Hint: In a right triangle with angles 30^o, 60^o the length of the side across the angle of 30^o is one-half the length of the hypotenuse.

Solution 1: Considering the circle and the circumscribed triangle as fixed, the "inside" triangle can be inscribed in many ways, which differ by a rotation. However, one of these rotations gives a particularly useful point of view. The circle touches the circumscribed triangle at the middle of each of its sides. Connecting these three mid-points we obtain an equilateral triangle inscribed in the circle. Clearly, this triangle splits the "bigger" triangle into four identical (equilateral) triangles - in fact each of these triangles has the lengths of its sides equal to 1/2 the length of the side of the circumscribed triangle. In particular, the area of the inscribed triangle is 1/4 the area of the circumscribed triangle, i.e., has area 15/4.

Solution 2: Here are the steps for you to follow and solve the problem analytically. Given an equilateral triangle with the length of its sides a, determine the radii of the inscribed and circumscribed circles in terms of a; use, for example, the hint to obtain these relations. Once we know these relations we can relate (express one by the other) the areas of the two triangles in our problem.

4) If $x+y+z=5$, $xy+xz+yz=6$ and $xyz=4$, what are
a) $x^2+y^2+z^2$? b) $x^3+y^3+z^3$?

Solution 1: (a) Note that $(x+y+z)^2 = x^2+y^2+z^2+2(xy+xz+yz)$. Hence, $x^2+y^2+z^2 = 25-2(6) = 13$.

(b) Similarly,

$$\begin{aligned}(x+y+z)^3 &= x^3+y^3+z^3+3(x^2y+xy^2+x^2z+xz^2+y^2z+yz^2)+6xyz\\ &= x^3+y^3+z^3+3[xy(x+y)+xz(x+z)+yz(y+z)]+6xyz\\ &= x^3+y^3+z^3+3[xy(5-z)+xz(5-y)+yz(5-x)]+6xyz\\ &\qquad\qquad = x^3+y^3+z^3+15(xy+xz+yz)-3xyz.\end{aligned}$$

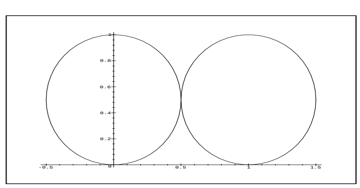
Thus $x^3+y^3+z^3 = 125-15\cdot 6+3\cdot 4 = 47$.

Solution 2: Most likely you cannot find a simpler proof of (a). However, there is another way to solve (b), which is worth noting. Take the three given numbers x, y and z and form the cubic polynomial $P(t) = (t-x)(t-y)(t-z)$. Expanding the product we can write $P(t)$ also as $P(t) = t^3-(x+y+z)t^2+(xy+xz+yz)t-xyz = t^3-5t^2+6t-4$ (this is how one can obtain the so called Viète's formulas relating the roots of a polynomial to its coefficients). Let S_k denote the sum of the k-th powers of x, y and z. Notice that by the definition of P, $P(x) = P(y) = P(z) = 0$, i.e.,

$$P(x) = x^3-5x^2+6x-4 = 0 \qquad P(y) = y^3-5y^2+6y-4 = 0$$
$$P(z) = z^3-5z^2+6z-4 = 0.$$

Adding the above three equation and solving for S_3 we find $S_3 = 5S_2-6S_1+12 = (5)(13)-(6)(5)+12 = 47$. Can you find S_4?

5) Suppose two circles of diameter one are tangent at the point $(\frac{1}{2},\frac{1}{2})$ and the first circle is tangent to the x-axis at $(0,0)$ and the second circle is tangent to the x-axis at the point $(1,0)$. What is the circle with the largest area that can be inscribed in the region bounded by the two circles and the x-axis?



Solution: Consider the right triangle formed by the center of one of the given circles, the center of the circle inscribed in the required region, and the common point of the two given circles. The latter is the vertex where the right angle is since the tangent to a circle is perpendicular to the segment connecting the center with the point of tangency. The radii of the two given circles are $R = 1/2$. Let r be the radius of the biggest circle inscribed in the required region. The right triangle we considered above has hypothenuse of length $R + r$, while the other two sides are, respectively, of lengths $R - r$ and R. From the Pythagororean theorem it follows $(R + r)^2 = (R - r)^2 + R^2$, hence, $R^2 + 2Rr + r^2 = R^2 - 2Rr + r^2 + R^2$. Solving for r we find $4Rr = R^2$. Since $R \neq 0$ we conclude $r = R/4$. Thus the biggest circle we can inscribe in the indicated region is four times smaller than the given circles, precisely, it is a circle of radius $1/8$.

6) Let f be a function defined on the integers as follows: $f(n) = n - 5$ for $n > 50$ and $f(n) = f(f(n + 6))$ for $n \leq 50$. What is the range of f?

   Solution: If the domain were $n > 50$, then the range is $f(n) > 45$. Note that $f(50) = f(f(56)) = f(51) = 46$. Also $f(49) = f(f(55)) = f(50) = 46$. Similarly, $f(n) = f(f(n + 6)) = f(n + 1) = 46$ for $n \leq 50$. Hence, the range is the set of integers greater than or equal to 46.

7) We start with the rectangle with vertices $(0, 0)$, $(3, 0)$, $(0, 4)$ and $(3, 4)$ and rotate it 90° clockwise about the vertex $(3, 0)$ forming a new rectangle. We again rotate this new rectangle 90° about the vertex $(7, 0)$ forming a new rectangle. Twice more we rotate 90° clockwise about the vertices $(10, 0)$ and $(14, 0)$. Find the area under the path formed by the movement of the point $(0, 1)$ under these rotations.

   Solution: The area under the curve is represented by summing the area of the triangle with vertices $(0, 0)$, $(0, 1)$ and $(3, 0)$, the area of the quarter circle of radius $\sqrt{3^2 + 1^2} = \sqrt{10}$, the area of the triangle with vertices $(3, 0)$, $(4, 3)$ and $(7, 0)$, the area of the quarter circle of radius $\sqrt{3^2 + 3^2} = \sqrt{18}$, the area of the triangle with vertices $(7, 0)$, $(10, 0)$ and $(10, 3)$, the area of the quarter circle of radius 3 and the area of the quarter circle of radius 1. Hence the area under the curve is $\frac{3}{2} + \frac{10\pi}{4} + 6 + \frac{18\pi}{4} + \frac{9}{2} + \frac{9\pi}{4} + \frac{\pi}{4} = 12 + \frac{19\pi}{2}$.

8) A group of 100 tourists arrived in a city. Of these tourists, 15 knew neither French nor Italian, 65 knew French, and 77 knew Italian. How many tourists knew both French and Italian?

   Solution: Let F be the set of people who knew French and I be the set of people who knew Italian. We are given that $100-15 = 85$ people know French or Italian. Partitioning this set of 85 people into three non-intersecting sets (groups of people), namely, the sets of people who know only French, the set of people who know only Italian, and the set of people who know both languages we see that the latter set has $65 + 77 - 85 = 57$ people. Alternatively, we can say this argument invoking the inclusion/exclusion principle $|F \cup I| = |F| + |I| - |F \cap I|$, where for a set S, $|S|$ denotes the number of elements in the set S. Hence, the number who knew both is $|F \cap I| = 65 + 77 - (100 - 15) = 57$.

   It is useful to represent the sets F and I by ovals which overlap. The common area will represent the people who know both French and Italian, while the remaining part of each of the sets will represent the people who know exactly one language - either French or Italian. The picture might help you "see" the above solution.

9) Suppose $x^2 - 6x + k$ and $x^2 - 10x + 2k$ have a common root. What must k be?

Solution: The roots of $p(x) = ax^2 + bx + c$ are obtained by setting $p(x) = 0$. Thus $k = 6x - x^2$ and $2k = 10x - x^2$. Multiplying the first equation by 2, we obtain $2k = 12x - 2x^2$. Therefore $12x - 2x^2 = 10x - x^2$, i.e., $2x - x^2 = 0$, which implies that $x = 0$ or $x = 2$. If $x = 0$, then $k = 0$. If $x = 2$, then $k = 8$. The given equations have a common root if $k = 0$ or $k = 8$.

10) A bank ATM machine has \$500 in one dollar bills. You are allowed to withdraw \$300 or deposit \$198. What is the maximum amount you can withdraw from this ATM machine? You can do any sequence of withdrawals and deposits (of the allowed type).

Solution: The key here is to notice that both the amount we can withdraw or deposit is divisible by six (and six is the greatest common divisor). Thus at any moment the amount we can hold in our hands is divisible by six. The biggest number, which is a multiple of six and is less than or equal to 500 is 498. Therefore this is the maximum amount we can possibly achieve, if it as achievable with consecutive withdrawals and deposits. We will show that this is indeed the case. Suppose at some point we have $\$x$ in our hand. We can either increase this amount to $\$(x + 300)$, provided $x + 300 \leq 500$, or decrease this amount to $\$(x - 198)$, if $0 \leq x - 198$. In particular, using withdrawals and deposits we can increase our amount by \$6 in the following way

$$x_1 = x + 300, \qquad x_2 = x_1 - 198 = 102 + x, \qquad x_3 = x_2 + 300 = 402 + x$$
$$x_4 = x_3 - 198 = 204 + x, \qquad x_5 = x_4 - 198 = 6 + x.$$

In the above process each of the x_i's is non-negative, so the corresponding deposit or withdrawal is "allowed", provided $402 + x \leq 500$, i.e., $x \leq 98$. Starting with $x = 0$ and performing the above round of deposits/withdrawals sixteen times we end up holding in our hand \$96. One more withdrawal of \$ 300 gives a total of \$396. At this point if we deposit \$198, and then withdraw \$300 we end up having \$498. This proves that the maximum amount we can withdraw from the ATM is \$498.

Comment: As we already observed, the greatest common divisor of 300 and 198 is 6. Can you write 6 as a linear combination with integer coefficients of 300 and -198, i.e., find integers n and m, such that, $6 = 300n - 198m$? In our problem n and m had to be positive integers (and satisfy even further restrictions), but the point to think about here is given two integers, can you write their greatest common divisor (gdc) as a linear combination of the two given numbers with integer coefficients? If you divide each of the numbers by the gcd you can reduce to having two numbers with gcd equal to 1. A possible way to obtain such representation is to use Euclidean algorithm.

Do not hesitate to contact us if you have any questions regarding the solutions (send us yours!) or you are interested in going a little further as suggested by the comments/ questions in the above solutions.