**Solutions for UNM–PNM Math Contest questions, Fall 2006**

The New Mexico math team has kindly provided us with beautiful solutions to the questions from the fall 2006 UNM–PNM math contest which are posted in a separate file. Here I will go through my own solutions, many of which are similar to those found by students.

**1.** For part (a), we find, by trial division, that $2006 = 2 \cdot 17 \cdot 59$. For part (b) write

$$x^2 - y^2 = (x - y)(x + y).$$

Since $x$ and $y$ are whole numbers so are $x - y$ and $x + y$ and so both $x - y$ and $x + y$ must be made up of the factors $2, 2, 17, 59$ of 4012. Now $x - y$ and $x + y$ have the same parity (that is they are either both even or both odd). Since the product is even, they must both be even. We also have $x + y > x - y$ and so there are two possibilities. Either we have $x - y = 34$ and $x + y = 118$ or we have $x - y = 2$ and $x + y = 2006$. These lead to two solutions for the equation, namely $x = 76, y = 42$ and $x = 1004, y = 1002$.

**2.** For the anagrams of "math," there are 4 places to put the 'm' and then three remaining places to put the 'a' and then two places to put the 't' and finally one remaining place for the 'h.' These anagrams are all different since the 4 letters are distinct. Multiplying these gives 24 possible anagrams of "math."

Next for Mississippi, the analysis is a little more complex but follows the same lines. For the 'm' there are 11 different places to put it. Next for the two p's there are 10 places to put the first, 9 for the second but we must divide this total by two because the 'p's are themselves indistinguishable: in other words we will not be able to tell the difference between putting the first p in place of the 'M' and the second p in place of the first 'i,' on the one hand, and placing the first p in place of the first 'i' and the second p in place of the 'M.' So this gives 45 possibilities for placing the two p's. Next for the four i's. There will be 8 places left for the first, 7 for the second, 6 for the third, and 5 for the fourth. As with the p's this counts indistinguishable cases many times, 24 times to be exact as there are 24 different ways to place the 4 i's in 4 determined places of Mississippi. Lastly for the s's, happily there is no choice left as there are only 4 places now open and all of them must be filled with s's. Counting up our choices we find that there are

$$\frac{11 \cdot 45 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{24} = 34650.$$

**3.** Suppose you place two points $P_1$ and $P_2$ on circle of radius one so that they are one unit distance from one another. Next form the triangle between $P_1$, $P_2$ and the center $P$ of the circle. Since the circle has radius one unit the lines joining $P$ and $P_1$ and then $P$ and $P_2$ also have length one. Thus this is an equilateral triangle and the angle $P_1PP_2$ is thus 60 degrees. This means that if we continue around the circle, placing a point $P_3$

at a distance one unit from $P_2$ (in the opposite direction to $P_1$) and so on, we will fit exactly 6 points on the circle, no two closer than one unit to one another. This gives a total of 7 points, counting the center $P$, which are pairwise at least one unit apart from one another. This is the largest number possible (which you should try to prove for yourself).

**4.** Many years ago, we were taught an algorithm for extracting square roots but of course I was not expecting any one to remember this method on the spot. What I was hoping to find is what I would call "enlightened" guess and check. Let me explain: the first important thing to note is that if I take two numbers $1 < a < b$ then of course $a^2 < b^2$ and since $a < \frac{a+b}{2} < b$ we have in fact

$$a^2 < \left(\frac{a+b}{2}\right)^2 < b^2.$$

Moreover $\left(\frac{a+b}{2}\right)^2$ will be a little closer to to $a^2$ than to $b^2$: for example $1^2 = 1$, $1.5^2 = 2.25$ and $2^2 = 4$ (why is this always the case??).

So we have $2^2 = 4 < 7 < 9 = 3^2$. Thus $\sqrt{7}$ is 2 plus some decimal. As far as the first digit of the decimal goes, based on where 7 falls relative to 4 and 9, it will be at least 5 but is more likely to be a 6 or even possibly a 7. Checking shows that $2.6^2 = 6.76 < 7 < 7.29 = 2.7^2$. Based on where 7 falls relative to $2.6^2$ and $2.7^2$ one would guess that the next digit is roughly a 5. It turns out that $2.64^2 \sim 6.97 < 7 < 2.65^2 \sim 7.02$, leading us to believe that the next digit is probably a 5 or 6. One last calculation shows that it is a 5 and the answer to the problem is 2.645.

**5.** Let $f(x) = ax^2 + bx + c$ be the equation of the parabola. We are given that $f(0) = 1$, $f(1) = 4$ and $f(2) = 9$. Plugging the first of these equations into $f(x)$ tells us that $c = 1$. Using this information plus the equation $f(1) = 4$ says that

$$a + b + 1 = 4.$$

Similarly using the fact that $f(2) = 9$ gives

$$4a + 2b + 1 = 9.$$

Solving these simultaneous linear equations gives $b = 2$ and $a = 1$ so

$$f(x) = x^2 + 2x + 1.$$

**6.** The first car crosses the startline every 6/5 minutes while the slower car crosses the startline every 2 minutes. The smallest common multiple of these two numbers is 6 and therefore the two cars cross the startline together *every* 6 minutes.

For the second part, the first car is travelling 40 miles per hour faster than the second so it will lap the second car when it has travelled 2 miles (at 40 mph). This takes 3 minutes and thus the slower car gets lapped *every* 3 minutes.

**7.** The answer to all three questions is yes. For 6 squares you can take one two by two and five one by one squares and put them together to make a three by three square. For 7 squares, you can do three two by two squares and four one by one squares which fit together into a four by four square. Finally for 8 you can do one three by three and seven one by one squares which fit together to make a four by four square.

**8.** Thinking of 2006 as $2000 + 6$, when we raise this to the fifth power, all terms in the expansion will have a factor of 2000 in them except the last one, $6^5$. Any number with a factor of 2000 ends in three zeroes and so does not affect the last three digits. As to $6^5$ its last three digits are 776 and so 776 are the last three digits of $2006^5$.

As for $5^{2006}$ we can look at the last three digits of the first few powers:

$$5, 25, 125, 625, 125, 625, 125, \ldots$$

So, other than the first two, the odd ones end in 125 and the even ones in 625. We have an even power, 2006, so the answer is 625.