UNM-PNM STATEWIDE MATHEMATICS CONTEST XXXVIII

February 4th, 2006 SECOND ROUND SOLUTIONS

PROBLEM 1

Find all positive integers n less than 1000 with the following properties: the remainder when n is divided by 25 is 1, the remainder when n is divided by 7 is 1, and the remainder when n is divided by 4 is 1.

The number 337 *is not one of the numbers you are asked to find because it has remainder* 1 *when divided by* 4 *and by* 7*, but when divided by* 25*, it has remainder* 12 *instead of* 1*.*

ANSWER: $n = 1$, or $n = 701$.

SOLUTION: By hypothesis n has remainder 1 when divided by 25, 7, and 4, that is, there exist non-negative integer numbers p, q, and r, such that

$$n = 25p + 1 = 7q + 1 = 4r + 1.$$

This is the same as saying that 25, 7, and 4 divide $(n-1)$, and since 25, 7, and 4 do not share any divisor, all three numbers must be factors of $(n-1)$, that is,

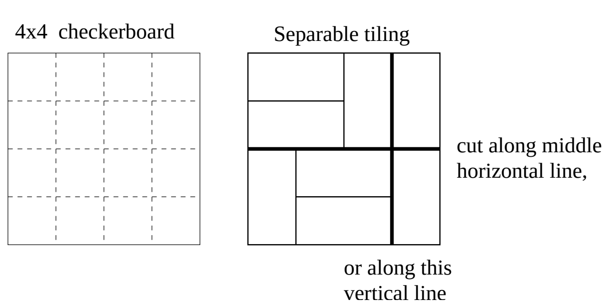
$$n - 1 = 25 \times 7 \times 4 \times k = 700 \times k, \quad \text{for some} \quad k = 0, 1, 2, \ldots$$

Since we are assuming $0 < n < 1000$, the only possible solutions are given by $k = 0$ or, $k = 1$; that is $n - 1 = 0$, or $n - 1 = 700$. This implies $n = 1$, or $n = 701$.

An alternative but much more laborious solution, is to list all numbers that have remainder 1 when divided by 4, 7, or 25, and that are smaller than 1000. This search can be made less cumbersome by observing that the numbers that have remainder 1 when divided by 25 must end in 26, 56, 76, or 01. Among those, the ones that will have remainder 1 when divided by 4 must be odd numbers, that is only those ending in 01 will work: 1, 101, 201, 301, 401, 501, 601, 701, 801, 901. Finally among those the only ones that have remainder 1 when divided by 7 are: 1 and 701.

PROBLEM 2

A four by four checkerboard with 16 squares can be covered exactly by 8 dominoes of two squares each, this is called a *tiling of the checkerboard by dominoes*. Likewise an n by n checkerboard can be tiled by $n^2/2$ dominoes provided n is even. Here is an example of a tiling of a four by four checkerboard.



Notice that it can be "cut" between two adjacent rows of the checkerboard (the middle ones!) without destroying any dominoes. Whenever a checkerboard tiled by dominoes can be cut by a horizontal or vertical line between some adjacent pair of rows or columns, we say that the tiling is *separable*, otherwise we say that the tiling is *non-separable*.

(a) Given a four by four checkerboard, can you find a non-separable tiling?

(b) Given a six by six checkerboard, can you find a non-separable tiling?

(c) Given an eight by eight checkerboard, can you find a non-separable tiling?

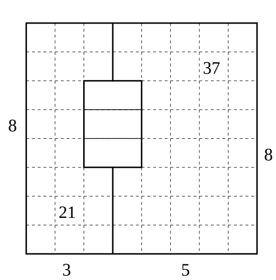
In all three cases, if your answer is YES, please show a tiling in the corresponding checkerboard provided in the Work Sheet.

ANSWER: (a) NO, (b) NO, (c) YES, see solutions.

SOLUTION: **(a)** In this case we can attempt to tile the four by four chessboard with the dominoes making sure every internal horizontal and vertical lines are cut at least once. We quickly observe that this task is impossible to accomplish. We can not start placing two vertical dominoes in the first column because the chessboard will be separable at the first vertical line, nor can we start placing 4 horizontal dominoes because the chessboard will be separable at the second vertical line. Therefore we must place one vertical and two horizontal dominoes in the first column. Without loss of generality, we can assume that the top is vertical and the two lower ones horizontal. Now consider the bottom row, we have one horizontal domino, we cannot place another horizontal domino because the chessboard will be separable by the third horizontal line, we must instead place two vertical ones, but then the chessboard is separable by the second horizontal line, and we have exhausted all our possibilities.

(b) We attempt to do the six by six case by hand as we did in the previous case. But it is more complicated since many more possibilities arise. One could attempt to write down all possibilities, but a good bit of patience is required. Instead we will try a different approach. To be non-separable each vertical and horizontal line must be cut at least once, and each domino cuts one and only one line. To be separable at least one line is not cut at all. We have 5 horizontal lines to be cut and 5 vertical ones. If all lines are to be cut, we need at least 10 dominoes, and we have 18 dominoes at our disposal, hmmmmm... The crucial observation is that each line is cut an EVEN number of times, so if it is cut is cut at least twice, and in this case, at most 4 times (if one line were cut 6 times then the tiling will be separable by the two boundary lines of the dominoes). But wait, if each line is cut at least twice, then we need at least 20 dominoes but we only have 18 dominoes, therefore not all lines are cut, and this means that all tilings are separable! We cannot find a non-separable tiling by dominoes of a six by six chessboard.

Why is it true that given a tiling of $2n$ by $2n$ chessboard, if a line is cut, it must be cut by an even number of dominoes?

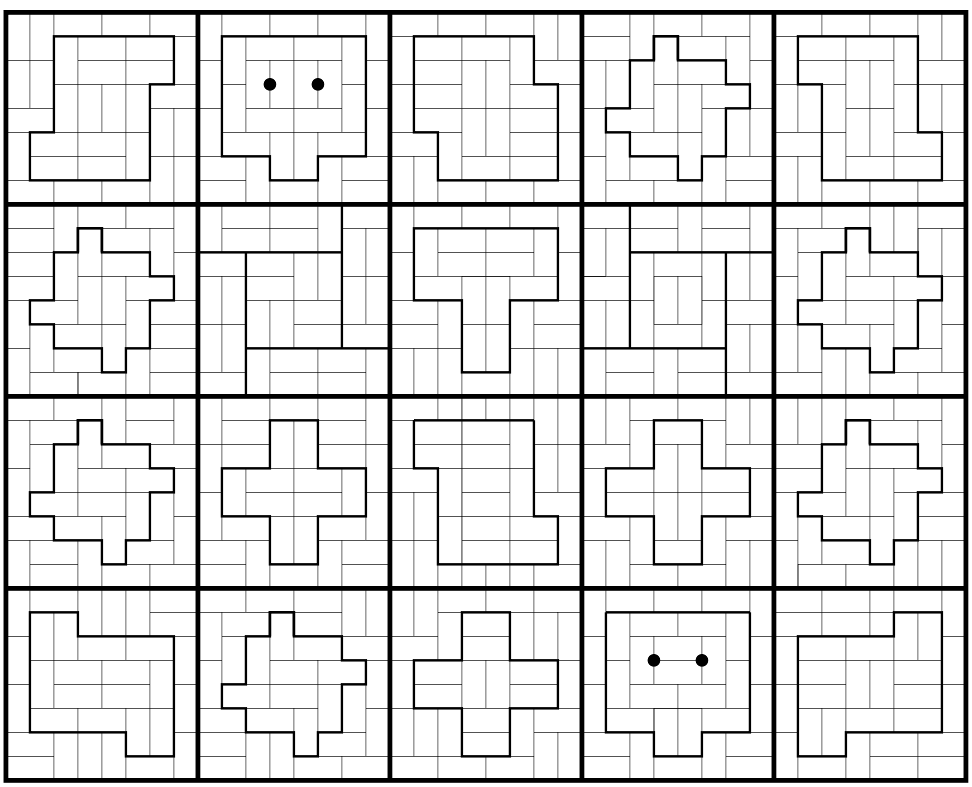


Suppose a line (for the sake of the argument assume is a vertical line) is cut by an odd number of dominoes, $2k+1$, then the chessboard will be divided in two pieces that must be tiled independently by dominoes, each one of them consisting of an even number of squares minus $2k+1$ squares covered by the dominoes which are cutting the line. That is we are required to cover with dominoes two regions consisting of an ODD number of squares, this is impossible. In the picture we have an 8 by 8 chessboard and a vertical line cut by 3 dominoes, the regions are highlighted with the number of squares that need to be covered.

Let us revisit part (a) in the light of these discoveries. In the case of four by four chessboard, there are 6 lines to be cut at least twice if we expect to encounter a non-separable tiling. This means we need at least 12 dominoes, but we only have 8 dominoes, therefore not all lines are cut, and this means that all tilings are separable! We cannot find a non-separable tiling by dominoes of a four by four chessboard.

(c) In the case of eight by eight chessboard, there are 14 lines to be cut at least twice if we expect to encounter a non-separable tiling. This means we need at least 28 dominoes, but this time we have 32 dominoes at our disposal. So there could be non-separable tilings, in fact if there were then most lines will be cut exactly twice but some could be cut 4 times or even 6 times. More precisely, any non-separable tiling will be such that 12 lines are cut twice and the remaining two are cut four times, or 13 lines are cut twice and the remaining one is cut 6 times. How to find such tilings? Almost anything one tries works. We have collected a number of symmetric tilings which were offered as solutions in the exam, we saw in these tilings crosses, spirals, skulls, zetas, T's, etc. We composed a "quilt" with them. I think only one of the tilings is not symmetrical, all others have at least one axis of symmetry. Find the asymetric tiling and the axis of symmetry of the others!

The difficult questions will be: how many truly different symmetric non-separable tilings can be found? How many non-symmetric ones? Truly different means, you cannot go from one to the other by rotation, or by reflection along some line, or point.



Crosses - Tony Huang (9th grade, La Cueva HS), Isaac Luhne (10th grade, La Cueva HS).
T - Adam Nekimker (9th grade, Los Alamos HS).
Skulls - Adam Izraelewitz (9th grade, Los Alamos HS), Nathan Hungate (9th grade, Del Norte HS).

Zeta's - Kevin rosillon (9th grade, El Dorado HS), Mallay McCampbell (9th grade Manzano HS), Nephi Lott (10th grade, Farmington HS).
Spirals with 4×4 **square core** - Julia Eichel (9th grade, El Dorado HS), Mary Coller (10th grade, El Dorado HS), Colleen Lanza (10th grade, El Dorado HS), Kristina Bagnell (11th grade, La Cueva HS), Alex Christensen (11th grade, Los Alamos HS).
Spirals with 2×2 **square core** - Andrew Chae (9th grade, Manzano HS),

PROBLEM 3

Find all pairs (x, y) of real numbers such that

$$x^{2004} + y^{2004} = x^{2005} + y^{2005} = x^{2006} + y^{2006}.$$

ANSWER: The pairs are $(1,1)$, $(1,0)$, $(0,1)$, or $(0,0)$.

SOLUTION 1: Obvious solutions are any combination of ones and zeroes, since neither one nor zero changes when raised to a power:

$$(0,0), (0,1), (1,0), \text{ or } (1,1).$$

THESE ARE ALL SOLUTIONS, but this needs to be justified!

It should be clear that if $x = 1$ and the equations hold then

$$1 + y^{2004} = 1 + y^{2005} = 1 + y^{2006},$$

hence $y^{2004} = y^{2005} = y^{2006}$, at this point either $y = 0$, or not, in which case we can divide by y^{2004} to get $1 = y = y^2$, hence if y is not zero it must be one. Similarly, if $x = 0$ then y must be 0 or 1. And by exactly the same argument if $y = 0$ or 1, then x must be equal to 0 or 1.

Let us assume that neither x nor y are 0 or 1, and they satisfy the above equations. These equations imply (by collecting x's on one side, and y's on the other)

$$x^{2004}(1 - x) = y^{2004}(y - 1),$$
$$x^{2004}(1 - x^2) = y^{2004}(y^2 - 1).$$

Since we are assuming that neither x nor y are 0 or 1, we can divide the second equation by the first[1] and we get

$$\frac{1 - x^2}{1 - x} = \frac{y^2 - 1}{y - 1},$$

which implies that $1 + x = 1 + y$, hence $x = y$. This only says that if x, y are neither 0 nor 1 and they satisfy the inequalities they might be equal. We still have to check back into the inequalities. Assume $x = y$, then $2x^{2004} = 2x^{2005} = 2x^{2006}$, and since x is not zero, it implies $x = 1$, but we assumed that $x \neq 1$, therefore there is no solution of the type $x = y$ other than the ones we already know, $x = y = 0$ or 1.

Notice that each equation has infinitely many solutions. Take for example the first equation,

$$x^{2004} + y^{2004} = x^{2005} + y^{2005},$$

[1] As Prof. Hahn says this is a reminder of the *cardinal law of mathematics*:
"Thou shall not divide by zero."

or consider the more general equation for $n \geq 1$.

$$x^n + y^n = x^{n+1} + y^{n+1}, \tag{1}$$

that will encompass both equations by setting $n = 2004$, and $n = 2005$.

Again we have the trivial solutions, but if we assume that $y \neq 0$, and introduce the new variable $t = x/y$, then the equation becomes,

$$t^n + 1 = y(t^{n+1} + 1),$$

which can be solved for y provided $t^{n+1} + 1 \neq 0$ (if $t^{n+1} = -1$, this implies that n is even, therefore $t^n = 1$ and the equation becomes $2 = 0$ which is impossible),

$$y = \frac{t^n + 1}{t^{n+1} + 1}.$$

Given any value of t so that $t^{n+1} + 1 \neq 0$, then the previous formula gives the value for y, and we can then obtain the value for x multiplying y by t,

$$x = t\frac{t^n + 1}{t^{n+1} + 1}.$$

Clearly if we use these formulas for x and y we get a solution of equation (1), in fact,

$$\begin{aligned} x^n + y^n &= \left(t\frac{t^n + 1}{t^{n+1} + 1}\right)^n + \left(\frac{t^n + 1}{t^{n+1} + 1}\right)^n \\ &= \left(\frac{t^n + 1}{t^{n+1} + 1}\right)^n (t^n + 1) \\ &= \frac{(t^n + 1)^{n+1}(t^{n+1} + 1)}{(t^{n+1} + 1)^{n+1}} \\ &= \left(\frac{t^n + 1}{t^{n+1} + 1}\right)^{n+1} + \left(t\frac{t^n + 1}{t^{n+1} + 1}\right)^{n+1} \\ &= y^{n+1} + x^{n+1}. \end{aligned}$$

We have an infinite number of solutions to equation (1) for each given $n \geq 0$, parametrized by $t \geq 0$. For example if $t = 1/2$, then $x = \frac{1+2^n}{1+2^{n+1}} < 1$, and $y = \frac{2+2^{n+1}}{1+2^{n+1}} > 1$. Notice that if $0 < t < 1$ then $y > 1$ and $x < 1$, and if $t > 1$ then $y < 1$ and $x > 1$.

Some students analyzed the original equations and noticed that it was not possible for x and y to be simultaneously larger than one or smaller than one. But that did not ruled out the possibility of x or y being larger than one and the other smaller.

SOLUTION 2 (11th graderAndrew Gu at Lakeside School, Seattle, WA): The two given equalities are,

$$\begin{aligned} x^{2006} + y^{2006} &= x^{2004} + y^{2004} \\ x^{2005} + y^{2005} &= x^{2004} + y^{2004}. \end{aligned}$$

Subtract twice the second equation from the first one and collect all terms on the left-hand-side to get,

$$x^{2006} + y^{2006} - 2\big(x^{2005} + y^{2005}\big) + x^{2004} + y^{2004} = 0.$$

Now collect the x-terms and the y-terms, and factor out the 2004 power of x and y respectively, to get,

$$x^{2004}(x^2 - 2x + 1) + y^{2004}(y^2 - 2y + 1) = 0.$$

The quadratic terms are perfect squares, so that zero is the sum of two non-negative quantities,

$$0 = x^{2004}(x-1)^2 + y^{2004}(y-1)^2,$$

therefore each summand must be identically equal to zero,

$$x^{2004}(x-1)^2 = 0, \qquad y^{2004}(y-1)^2 = 0.$$

These equations hold if and only if $x = 0$ or $x = 1$, and $y = 0$ or $y = 1$. This shows that the only possible solutions to the equations are the ones found by inspection.

PROBLEM 4

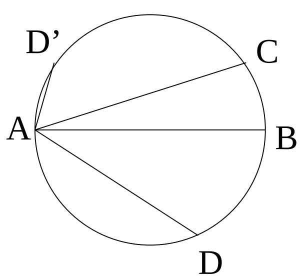
Suppose A, B, C, D are four points on the same circle such that the length of AB is 5 units, and the angle CAD is $\pi/3$ (we will use the notation: $|AB| = 5$, and $\angle CAD = \pi/3$).

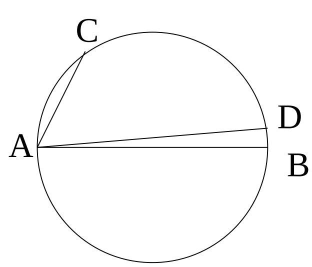
(a) Assume AB is a diameter of the given circle. What is the length of CD?

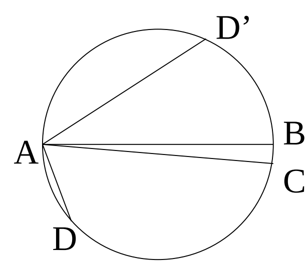
(b) Assume instead that $\angle ACB = \pi/4$ (notice that AB is NOT a diameter of the circle in this case, why?). What is the length of CD?

ANSWER: (a) $|CD| = 5\sqrt{3}/2$, (b) $|CD| = 5\sqrt{6}/2$.

SOLUTION 1: (a) The first observation is that there are infinitely many possible configurations. In fact, select a diameter with endpoints A, and B. Select any point other than A on the circle an name it C. Then there exist one or two points on the circle, D and perhaps D', such that $\angle CAD = 60°$ and $\angle CAD' = 60°$.[2]





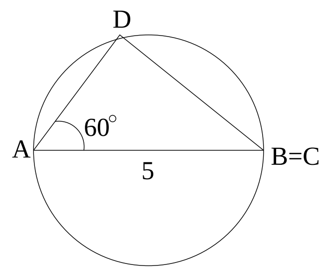


From the pictures it is not completely obvious that given C, $|CD| = |CD'|$, although this is true. We are asked to find $|CD|$, and the way the question is presented, it seems to imply

[2] It was pointed out to us during the solution session for parents and teachers that the younger students might not be aware of the notion of π and measuring angles in radians. We will try to remember in the future to use instead of π, 180°.

that such length is independent of the choice of C. This is also true, let us postpone for a moment the proof of these facts.

Assuming that we know the lenght $|CD|$ is independent of the location of the points C and D, it only depends on the angle CAD, then we can choose them to be wherever we please.



The simplest option is to let $C = B$, and then the $\triangle ADB = \triangle ADC$ is a right triangle, with hypothenuse the diameter AB that has length 5 units, $\angle CAD = 60°$, therefore the length of opposite side $DB = DC$ can be calculated,

$$|CD| = |AB| \sin 60° = \frac{5\sqrt{3}}{2} \text{ units.}$$

A different argument, still in the case $C = B$, can be used to compute $|CD|$. Let O be the center of the circle, then $|OA| = |OB| = |OD| = r = |AB|/2 = 5/2$ units. We have two isosceles triangles $\triangle AOD$ and $\triangle BOD$, with common angles,

$$\angle ADO = \angle DAO = 60°, \quad \angle ODB = \angle OBD.$$

Because the sum of the angles of a triangle is always 180°,

$$\angle DOB = \pi - \angle AOD = \pi - (\pi - 2\angle DAO) = 2\angle DAO = \angle 120°.$$

See the geometric fact below (and the companion picture), this duplication of the angle is not a coincidence.

Given the isosceles triangle $\triangle DOB$, with the equal sides of a known length ($|OD| = |OB| = 5/2$), and the angle between those sides is known ($\angle DOB = 120°$), then one can always find the length of the other side. For example, one can use the *Law of the Cosines*,

$$|BD|^2 = |CD|^2 = |OD|^2 + |OB|^2 - 2|OD| \times |OB| \cos\angle DOB = 2 \times \frac{25}{4}\left(1 + \frac{1}{2}\right) = \frac{75}{4},$$

therefore $|CD| = \sqrt{\frac{75}{4}} = \frac{5\sqrt{3}}{2}$.

Instead of the Law of the Cosines (which really did not take advantage of the fact that we had an isosceles triangle), let H be the midpoint of $CD = BD$, then OH is perpendicular to CD because the triangle is isosceles, and $\angle HOD = \frac{\angle DOB}{2} = 60°$, now we have a right triangle, say $\triangle OHD$, then

$$|HD| = |OD| \sin 60° = \frac{5}{2}\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{4}.$$

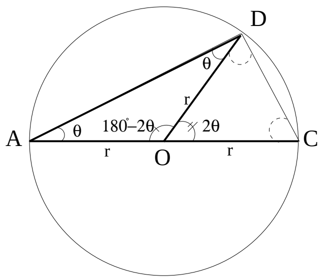
Now we are done, because $|CD| = 2|HD| = \frac{5\sqrt{3}}{2}$.

Many students worked out this particular case and got the right answer, but it was not clear at all that they knew that the answer to the particular case gave the answer in general.

Why is it true that $|CD|$ is independent of the location of C and D?

Geometric Fact: *Given a circle with center O, let A, C and D be three different points on the circle, let $\theta = \angle CAD$, then $\angle COD = 2\theta$.*

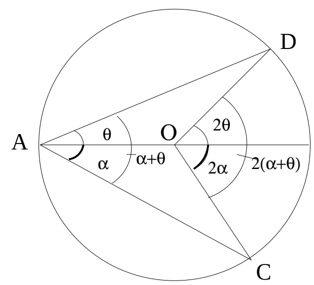
We already used this fact in the particular case AC is a diameter and $\theta = 60°$ a few paragraphs above.



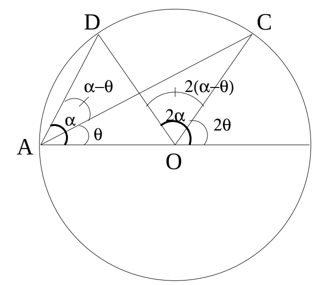
Consider first the case when AC is a diameter, then the same picture used in the first round to verify that $\angle ADC = 90°$ allows as to verify that $\angle COD = 2\theta$. We are using that the sums of the angles of a triangle is $180°$, and that $\angle AOD + \angle DOC = 180°$.

When AC is not a diameter, consider the diameter determined by the segment AO. Two cases arise, either the AO is inside $\triangle ACD$, or is outside. In either case we can use our special case to conclude the desired result, instead of writing the details, we just draw two pictures.

Case 1:



Case 2:



Given an arc on a circle, defined by points C and D (notice that each pair of points defines two arcs, we are choosing one of them, the other is called the *supplementary arc* to the given one), then $\angle CAD$ is the same for all points A on the supplementary arc to the given one, because they are all equal to half the angle to the center $\angle COD$ which is independent of the location of the point A. Such angle is called *the angle subtended by the arc* CD. Similarly given a point A on the circle, and an angle θ with vertex on A, let C and D be the intersection points of the legs of the "angle" with the circle, then $|CD|$ is independent of the position of the legs, it only depends on the angle θ. In fact, if r is the radius of the circle,

$$|CD| = 2r\sin\theta. \tag{2}$$

(Draw your own pictures to see what is going on!)

Notice that if θ is the angle subtended by arc CD, then the angle subtended by its supplementary arc, is exactly $180° - \theta$, the supplementary angle to θ. Why?

Notice also that if the angle subtended by arc CD is $90°$, then the chord CD must be a diagonal of the circle!! And if the chord CD is a diameter then the angles subtended by the chord CD and its supplementary arc are $90°$. Why?

(b) Armed with the knowledge gathered in part (a), we can now prove this part fairly quickly. This time we know that $|AB| = 5$ units, but AB is not a diameter, because $\angle ACB = \alpha = 45°$ (had AB been a diameter then $\angle ACB = 90°$, which is not by assumption). We also know that $\angle CAD = \theta = 60°$ as before, therefore formula (2) is valid, except that this time the radius r of the circle is not given. However we are given other information and we should be able to use that information to compute the radius. In fact the same formula shows that

$$|AB| = 2r\sin\alpha \quad \Rightarrow \quad r = \frac{|AB|}{2\sin\alpha},$$

and substituting the values of $|AB|$ and α we get that $r = \frac{5}{\sqrt{2}}$. We can now find the desired length,

$$|CD| = 2r\sin\theta = 2\times\frac{5}{\sqrt{2}}\times\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{\sqrt{2}} = \frac{5\sqrt{6}}{2} \text{ units.}$$

Notice that in this case there are also a number of possible configurations, you have the freedom to choose C on the longest arc determined by AB, and once you have chosen C then you can have one or two choices of D, but as explained before, regardless of these choices the length of $|CD|$ is constant. Many students found the correct length by considering again a particular and favorable configuration. A good choice is to let C be such that AC is a diameter, this forces $\angle ABC = 90°$, and it is then immediate that $|AC|/2 = r = |AB|/\sqrt{2} = 5/\sqrt{2}$. By the same token, $\angle ADC = 90°$, it follows that

$$|CD| = |AC|\sin 60° = 2\times\frac{5}{\sqrt{2}}\times\frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{\sqrt{2}}.$$

Solution 2 (Prof. Hahn): Prof. Hahn suggested part (b) of this problem. What he had in mind was a problem that will correctly teach the *Law of the Sines*. Given a triangle with angles α, β, γ, and the length of the opposite sides to the angles are a, b and c respectively, then the Law of the Sines says,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2r,$$

where r is the radius of the circumscribed circle.

Most students will remember the first two equalities in the law, but will forget the last one. The proof of this law is already done in part (a), in particular formula (2).

If you remember this law, then you can apply it to solve the problem.

(a) In this notation, we are given $A = \angle CAD = 60°$, we are given the radius $r = |AB|/2 = 5/2$, and we want to find $a = |CD|$. By the Law of the Sines, $a = 2r\sin 60° = 5\sqrt{3}/2$.

(b) Is basically what we did in the previous proof. Instead of appealing to (2) twice, we appeal to the Law of the Sines twice.

PROBLEM 5

Denote by $[x]$ the greatest integer not greater than x. For example: $[13.41] = 13$, $[54] = 54$, $[-3.12] = -4$.

(a) Is there a real number x such that

$$[x] + [2x] + [4x] + [8x] + [16x] + [32x] = 2006 \ \ ?$$

If your answer is YES, find the smallest such number.

(b) What if 2006 is replaced by 2005?

ANSWER: (a) $x = 31.875$, (b) There is no solution.

SOLUTION 1: Let n denote the integer part of x.

What is the integer part of $2x$? It is either $2n$ or $2n+1$, because $x = n+d$ where $0 \leq d < 1$ is the decimal part of x, and $2x = 2n + 2d$, if $2d < 1$ then $[2x] = 2n$, and if $2d \geq 1$ ($2d < 2$) then $[2x] = 2n + 1$.

How about the integer part of $4x$? Well, by the same token, it must be $2[2x]$ or $2[2x] + 1$, in terms of n there are four possibilities: $4n$, $4n + 1$, $4n + 2$, $4n + 3$ (the first two cases arise when $[2x] = 2n$, the last two when $[2x] = 2n + 1$). It should be more or less clear that the integer part of $8x$ should be of the form $8n + k$, for $k = 0, 1, 2, 3, 4, 5, 6, 7$, depending on what $[4x]$ was, etc, etc.

We wish to keep track of all the integer parts of the multiples of x involved in the equation. Any $n \leq x < n + 1$ has a unique representation of the form

$$x = n + \frac{a}{2} + \frac{b}{4} + \frac{c}{8} + \frac{d}{16} + \frac{e}{32} + f,$$

where a, b, c, d, e are either 0 or 1, and $f < 1/32$. Moreover,

$$\begin{aligned}
[x] &= n, \\
[2x] &= 2n + a \ (= 2[x] + a), \\
[4x] &= 4n + 2a + b \ (= 2[2x] + b), \\
[8x] &= 8n + 4a + 2b + c \ (= 2[4x] + c), \\
[16x] &= 16n + 8a + 4b + 2c + d \ (= 2[8x] + d), \\
[32x] &= 32n + 16a + 8b + 4c + 2d + e \ (= 2[16x] + e).
\end{aligned}$$

Notice that because $f < 32$, then f contributes nothing to the integer parts. In the worst case $32f < 1$. Therefore we have the freedom to choose $0 \leq f < 1/32$, the sum of the integer parts of all the x's in the above form for fixed a, b, c, d, e is the same regardless of the value of f. The smallest such x will be the one such that for given a, b, c, d, e the equation holds, and $f = 0$.

(a) Our hypothesis says that if we add up the equalities we get 2006, with this notation the right hand side becomes,

$$\begin{aligned}
2006 &= (1 + 2 + 4 + 8 + 16 + 32)n + (1 + 2 + 4 + 8 + 16)a \\
&\quad + (1 + 2 + 4 + 8)b + +(1 + 2 + 4)c + (1 + 2)d + e \\
&= 63n + 31a + 15b + 7c + 3d + e.
\end{aligned}$$

We can already compute n. Divide 2006 by 63 and the result will be 31 with remainder 53, that is $2006 = 31 \times 63 + 53 = 1953 + 53$. It should be clear that $n = 31$, it cannot be larger because $32 \times 63 = 2016 > 2006$, and it cannot be smaller because $2006 - 63n \leq 31 + 15 + 7 + 3 + 1 = 57$ (this is the case $a = b = c = d = e = 1$), and $2006 - 30 \times 63 = 116 > 57$. Hence, there will be a solution to the equation provided we can solve the equation

$$53 = 31a + 15b + 7c + 3d + e,$$

for some choice of a, b, c, d, e equal to zero or one. We can certainly do that, set

$$a = b = c = 1, \qquad d = e = 0.$$

The solutions of the equation are of the form,

$$x = 31 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + f = 31.875 + f, \quad \text{where} \quad 0 \leq f < 1/32.$$

In other words the solutions x to the equation belong to the interval

$$[31.875, 31.875 + 0.03125) = [31.875, 31.90625).$$

The smallest solution is $x = 31.875$.

(b) This time when we add up the equalities for the integer parts we get,

$$2005 = 63n + 31a + 15b + 7c + 3d + e.$$

In this case $n = 31$, but the reminder is 52 instead of 53. There will be solutions to the equation if and only if there are solutions to the equation

$$52 = 31a + 15b + 7c + 3d + e,$$

for some choice of a, b, c, d, e equal to zero or one. But this time there are no solutions to this equation. The next number below 53 that we can reach is 50, when setting $a = b = d = e = 1$ and $c = 0$, and we cannot do better.

There are no solutions to the equation with 2005.

PROBLEM 6

(a) You are given 13 points on a circle equally spaced. Suppose each point is colored either red or blue. Can you always find three points of the same color that are vertices of an isosceles triangle?

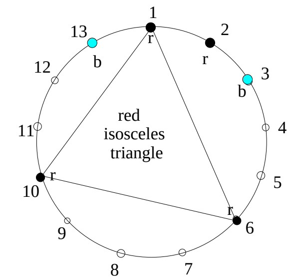
(b) What is the minimum *even* number of equally spaced points on a circle that guarantees some three points of the same color are vertices of an isosceles triangle (if each point is colored either red or blue)?

ANSWER: (a) YES, you can always find a monochromatic isosceles triangle.

(b) The minimum even number is 10.

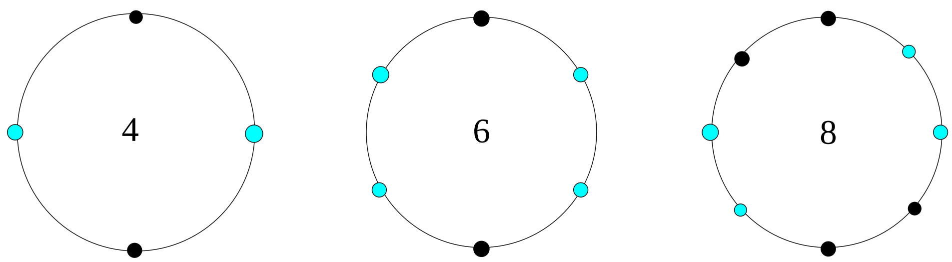
SOLUTION 1 (Prof. Hahn): **(a)** Observe that because we have 13 points, there must be two consecutive points of the same color, say red for the sake of the argument (this is true whenever we are given an ODD number of points). Notice also that if we have 3 consecutive points of the same color, we have found a monochromatic isosceles triangle. Therefore the neighbors to the two consecutive red points must be blue.

Let us number the points clockwise, so that points numbered 1 and 2 are consecutive red points, points 13 and 3 are blue. Now we have two blue points separated by two points, if we want to avoid a blue isosceles triangle, then points numbered 6 and 10 must be red. But then the triangle determined by points numbered 1, 10 and 6 is red and it is isosceles at vertex 10.



Therefore we cannot avoid having a monochromatic isosceles triangle given 13 points on the circle. However given an arbitrary odd number of points, it is clear that an argument like this should work, but perhaps several iterations are needed before forcing a monochromatic isosceles triangle into the picture. See Solution 2 which works for all odd number of points!

(b) In the case of an even number of points, we need at least 4 points. The pictures below present colorings where NO monochromatic isosceles triangle can be found for 4, 6 or 8 equally spaced points on the circle,

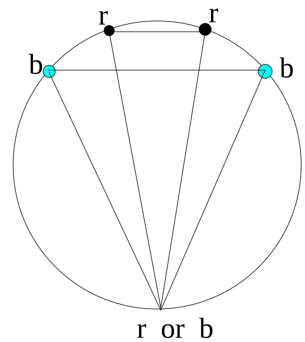


When you try to find a similar example in the case of 10 points, it does not work. Again we do not want to allow three consecutive points colored the same way, because that creates a monochromatic isosceles triangle. So we have at most two consecutive points of the same color as in the previous case. However this time there exists the possibility of not having two consecutive points of the same color, but the only possible such configuration is the alternating colors configuration which includes many monochromatic triangle of both colors. We are left to analyze the case when there are two consecutive points of the same color, say red, and as before say the are points numbered 1 and 2. As in the case of 13 points, since we are trying to avoid monochromatic triangles, we are forced to color blue the points numbered 10 and 3. To avoid a blue monochromatic triangle, we are forced to color red the points numbered 6 and 7, which are consecutive, therefore their neighbors, points 5 and 8 must be colored blue. At this point only points numbered 4 and 9 are left to be colored, and no matter what color we choose they will create a monochromatic isosceles triangle. In fact, if we color any of them blue, then there will be three consecutive blue points, hence a blue isosceles triangle. If we color them red then we create at least four red isosceles triangles with vertices at the points numbered $(1,9,7)$, $(2,9,6)$, $(2,4,6)$. $(1,4,7)$. Therefore there is no way to avoid a monochromatic isosceles triangle when we have 10 points, and 10 is the least amount of EVEN points that have that property.

SOLUTION 2 (Presented by 9th graders Christopher Smith from Las Cruces HS, Kristin Cordwell, Sarah Rowe and Steven Benner from Manzano HS):

(a) This is what we call a *Solution from the book* (from Erdös' book of solutions!).

As in Solution one we are reduced to analyze the case when there are two consecutive points of the same color (say red), whose neighbors must be colored blue. The brilliant observation that all these students did was that the point opposite to them (in our case, following the same numbering as we used in Solution 1, the point will be numbered 8) defines two isosceles triangles, one with the consecutive red points, the other with the blue neighbors, so regardless of the color chosen a monochromatic isosceles triangle is created.



Notice that this argument works for no matter how many odd points we have, as long as we have at least 5 such points. Also notice that the argument works equally well as long as you have two points of the same color, (say red) regardless of them being adjacent or not. Instead of considering the immediate neighbors, you consider the equidistant neighbors, that is the two points so that the four points are equidistant, and they have to be blue, otherwise a monochromatic isosceles triangle has been constructed. Now consider the point that is equidistant to the two red points (there is one and only one such point because we have an odd number of points). This point is necessarily equidistant to the blue points too, so no matter how we color it we will create a monochromatic isosceles triangle.

PROBLEM 7

Let $ABCD$ be a convex quadrilateral with perimeter p. *Convex means that, given any two points inside $ABCD$, then the segment joining them is also inside $ABCD$.*

(a) Can you find a point M inside $ABCD$ so that the sum of the distances from M to the four vertices is equal to the sum of the lengths of the two diagonals? If your answer is YES, please describe the point in the Work Sheet.

(b) Let M be an arbitrary point inside $ABCD$. Show that the sum of the distances from M to the four vertices is greater than or equal to the sum of the lengths of the diagonals, but smaller than $3p/2$.

(c) Can you find a convex quadrilateral $ABCD$ of perimeter p and a point M inside it so that the sum of the distances from M to the four vertices is larger than $p \times 1.49$? If your answer is YES, please draw your example in the Work Sheet.

ANSWER: (a) YES, the intersection of the diagonals.
(b) See solutions for the proofs.
(c) YES, see solutions for the example.

SOLUTION: Assume the vertices of the quadrilateral are labeled counterclockwise in alphabetical order.

(a) YES, just let M be the intersection point of the diagonals AC and DB, then clearly

$$|AM| + |MC| = |AC|, \qquad |BM| + |MD| = |BD|.$$

Therefore,

$$S := |MA| + |MC| + |MB| + |MD| = |AC| + |BD|.$$

(b) If M is not the intersection point of the diagonals then the triangle inequality[3] implies that, applied to $\triangle AMC$ and $\triangle BAD$,

$$|AC| \leq |AM| + |MC|, \qquad |BD| \leq |BM| + |MD|.$$

Hence,

$$|AC| + |BD| \leq |MA| + |MC| + |MB| + |MD| = S.$$

This shows that S is greater or equal than the sum of the lengths of the diagonals, and equality can hold for no matter what quadrilateral we are considering as was shown in part (a), by letting M be the intersection point of the diagonals AC and DB.

How about an upper bound? We are told to show that S cannot be larger than 3/2 of the perimeter p, in fact it cannot be equal except in the case when the quadrilateral is *degenerate*, and there are no interior points. More precisely, consider the case when three vertices coincide, say $A = B = C$, and the fourth vertex and M coincide, in our case $M = D$, then

$$|MD| = 0, \quad |MA| = |MB| = |MC| = |AD|, \quad p = 2|AD|, \quad \text{and} \quad S = 3|AD| = 3p/2.$$

We will show that given a convex quadrilateral (with non-empty interior, that is the quadrilateral is not a degenerate flat one), then

$$\begin{aligned}
|AM| + |BM| &\leq |BC| + |CD| + |DA|,\\
|BM| + |CM| &\leq |CD| + |DA| + |AB|,\\
|CM| + |DM| &\leq |DA| + |AB| + |BC|,\\
|DM| + |AM| &\leq |AB| + |BC| + |CD|.
\end{aligned}$$

Furthermore, equality in each one of the inequalities only occurs in degenerate cases, by identifying two or more vertices, and identifying M with a vertex. Equality in all four cases occurs only in the case when three vertices coincide, and the fourth vertex and M coincide, which is the case we are proscribing (we want to have non-empty interior).

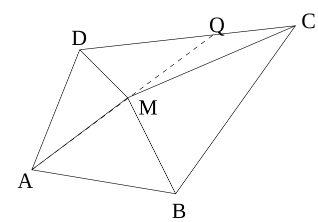
These will immediately prove the claim by adding up all four inequalities,

$$2(|AM| + |BM| + |CM| + |DM|) < 3(|AB| + |BC| + |CD| + |DA|),$$

hence $2S < 3p$, and $S < 3p/2$.

Suffices to verify just one of the above inequalities, the others follow by identical argument just relabeling the points. We will verify that $|AM| + |DM| \leq |AB| + |BC| + |CD|$.

Let Q be the intersection point of the line determined by AM and the boundary of the quadrilateral, in our picture, Q lies on the segment CD, but it could had been on the segment BC had C been closer to D. We assume for the sake of the argument that Q lies on CD, in the other case the argument is the same after relabeling.



[3]The length of a side of a triangle is smaller or equal to the sum of the lengths of the other two sides.

It is clear from the triangle inequality applied to $\triangle ABQ$ and $\triangle QBC$ that

$$|AQ| = |AM| + |MQ| \leq |AB| + |BQ| \leq |AB| + |BC| + |CQ|.$$

It is also clear that the triangle inequality applied to $\triangle DMQ$, that

$$|DM| \leq |MQ| + |DQ|.$$

Adding both inequalities we get,

$$|AM| + |MQ| + |DM| \leq |AB| + |BC| + |CQ| + |MQ| + |DQ|,$$

canceling the summand $|MQ|$ which appears on both sides, and noting that $|CQ| + |DQ| = |CD|$ we obtain the desired inequality,

$$|AM| + |DM| \leq |AB| + |BC| + |CD|.$$

(c) We already mentioned that we can achieve exactly $3p/2$ in the case of a degenerate quadrilateral where, for example, $A = B = C$ and $M = D$. Other than that, we get strict inequality, $S < 3p/2$.

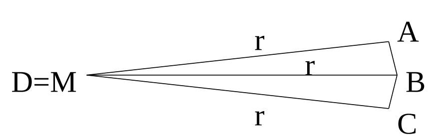
Given a fixed non-degenerate quadrilateral, we might be far from reaching the upper bound. For example, if we restrict ourselves to rectangles with sides of length $a \leq b$, then $p = 2(a+b)$, and the worst case scenario occurs when M is as far as possible from the intersection point of the diagonals, that is when M is equal to a vertex (why?). In this case

$$S = a + b + \sqrt{a^2 + b^2} \leq 2(a + b) = p.$$

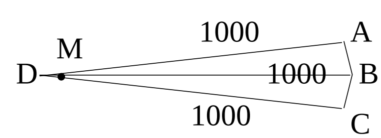
In the universe of rectangles, the upper bound is p which far from $3p/2$.

We must look beyond rectangles. Some students considered parallelograms with side-lengths $a \leq b$. The perimeter is the same as in the case of the rectangles, $p = 2(a + b)$. Let AC be the longest diameter, BD the shortest (in the case of the rectangles $|AC| = |BD|$ so there is no need for a distinction). As before the worst case scenario occurs when M is as far as possible from the intersection point of the diagonals, that is when M is equal to A or C. In this case, $S = a + b + |AC| \leq 2(a + b) = p$, again we are far from the upper bound $3p/2$.

The desired example is just a slight fattening of the degenerate case. Set A, B, C to be equidistant (distance $\epsilon > 0$) on the circle centered at D with large radius r. Let $M = D$, then $MA = MB = MC = r$, $MD = 0$, so $S = 3r$ and $p = 2r + 2\epsilon$. We need to choose ϵ compared to r so that $S > 1.49p$. We need $3r > 1.49 \times 2(r + \epsilon)$, so we need $\epsilon < 0.01r$. To be on the safe side, choose $\epsilon = 10^{-3}r$, then we can choose M very close to D but not identical to D, and the inequality will be satisfied. For example, one could take $r = 10^3 = 1000$,and $\epsilon = 1$, and M lying on the segment DB so that $|MD| = 1$.



|AB| = |BC| = ε



|AB| = |BC| = |DM| = 1

In this case $p = 2002$, $|DM| + |MB| = 1000$, and $|MA| = |MC| \geq 998$, so that

$$S \geq 1000 + 2 \times 998 = 2996 > 2982.98 = 1.49 \times p.$$

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Dear students: If you have any suggestions about the Contest, or if you have different solutions to any of this year's problems, please send them to:

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University of New Mexico
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or e-mail them to: *crisp@math.unm.edu*

Remember that you can find information about past contests at:
http://www.math.unm.edu/math_contest/contest.html

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