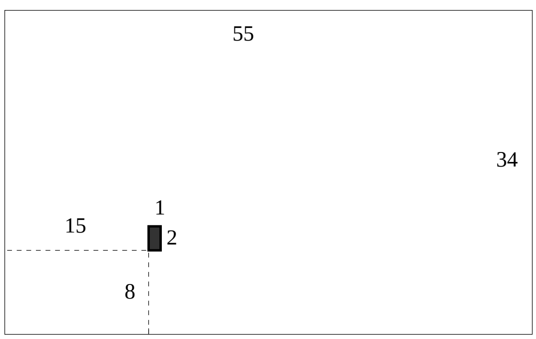
UNM-PNM STATEWIDE MATHEMATICS CONTEST XXXVIII

November 12, 2005 FIRST ROUND THREE HOURS

PROBLEM 1 Matthew is baking a cake. His recipe calls for: $1\frac{1}{3}$ cups of flour, $1\frac{3}{4}$cups of milk, and $1\frac{1}{12}$ cups of sugar. He has two measuring cups that he inherited from uncle Archimedes. They measure 2/3 of a cup and 1/4 of cup. Can he bake his cake? If YES, what is the smallest number of measurements needed to make this cake? Please indicate in the work sheet the number of times Matthew used each measuring cup to obtain the required amount of flour, milk and sugar.

PROBLEM 2 Builder Artemisa has to pave with tiles a rectangular patio with dimensions 34 units by 55 units. The patio has a rectangular fountain with dimensions 1 unit by 2 units that will not be paved. The exact location of the fountain is indicated in the picture below.



Artemisa's tile machine is pretty temperamental and *only produces square tiles of integer dimensions* (1 unit by 1 unit, 2 units by 2 units, etc.), moreover *once it produces a tile of a given size then it refuses to produce more of that size.* Fortunately Artemisa is very strong and she can handle square tiles of any dimension.

Will Builder Artemisa be able to cover her patio completely with square tiles of different sizes? (She is not allowed to break her tiles, pave beyond the patio, or to superimpose tiles). If the answer is YES, please show a paving in the map provided in the work sheet and write inside of each square tile its side length.

PROBLEM 3 You have a dart board divided in two regions, one red, one black. If you hit the red region you get 7 points if you hit the black region you get 10 points. Can you get 83 points? Can you get 22 points? What is the largest number of points you can NOT get?

PROBLEM 4

(a) We are given 6 points on a circle equally spaced (think of a clock with just the even hours). How many triangles can be constructed so that their vertices are three of the given points on the circle? How many among those triangles are: isosceles triangles? equilateral triangles? right triangles?

(b) Same question as above, except that now you chose among 2005 equally spaced points on the circle.

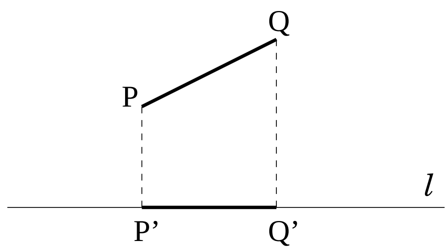
PROBLEM 5

(a) Can you find a positive integer k so that the first two digits of 2^k are 65?

(b) Can you find a positive integer n so that the first digit of 2^n is 7?

PROBLEM 6 We are given a line segment PQ and a triangle ABC in the same plane. Suppose the perpendicular projections of the segment PQ on the lines containing sides AB, BC, and CA, have lengths 3, 4, and 5, respectively. If the lengths of sides BC and CA are 6 and 3, respectively, find the length of the remaining side AB. Is the answer unique?

Note: *The* perpendicular projection *of a line segment PQ on a line ℓ is the segment $P'Q'$ on the line ℓ found by dropping lines perpendicular to ℓ from P and Q, see figure below,*



PROBLEM 7 What is the remainder when

$$P(x) = 1 - x + 2x^4 - 3x^9 + 4x^{16} - 5x^{25} + 6x^{36} + x^{2005}$$

is divided by $D(x) = x^2 - 1$?

Note: *The* remainder *is a polynomial $R(x)$ of degree smaller than the* divisor *$D(x)$ such that there is another polynomial $Q(x)$, the* quotient, *such that*

$$P(x) = Q(x)D(x) + R(x).$$

Given a polynomial $P(x)$ and a divisor $D(x)$ then the remainder and the quotient are uniquely determined. For example, if $P(x) = x^3 + 2x^2 - 4$, and we divide by $D(x) = x^2 - 1$, then the quotient and remainder are $Q(x) = x + 2$, $R(x) = x - 2$, since

$$x^3 + 2x^2 - 4 = (x + 2)(x^2 - 1) + (x - 2).$$

PROBLEM 8 We are given rectangles A and B with side lengths a_1, a_2 and b_1, b_2, respectively. Let C be the rectangle with side lengths a_1, b_2, denote its diagonal length c. Let D be the rectangle with side lengths a_2, b_1 denote its diagonal length d. Let R be the rectangle with side lengths c, d.

(a) Can the sum of the areas of A and B equal the area of R?

(b) Can the sum of the areas of A and B be strictly larger than the area of R?

(c) Can the sum of the areas of A and B be strictly smaller than the area of R?

If you answered YES to any of the above questions, please write down an example in the work sheet.