UNM-PNM STATEWIDE MATHEMATICS CONTEST XXXVII

November 13, 2004 FIRST ROUND SOLUTIONS

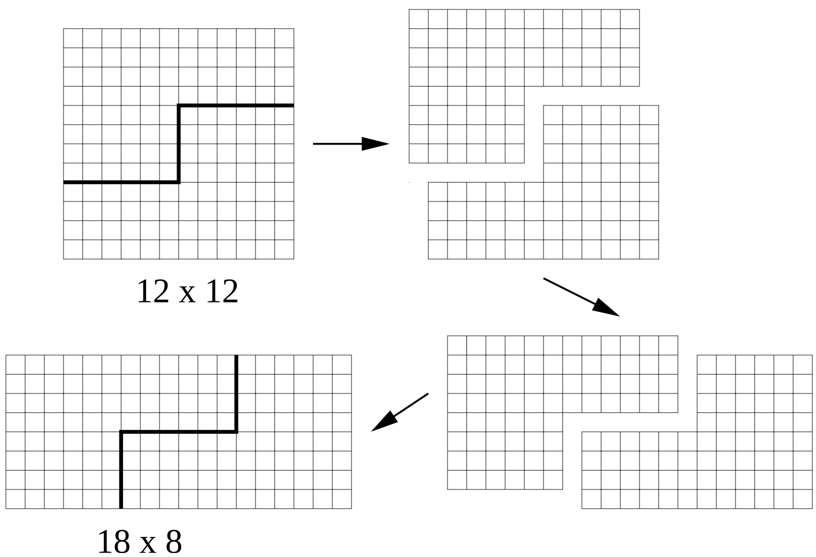
PROBLEM 1: You are given a 12×12 ft^2 carpet.

(a) Can you cut the 12×12 carpet into two (2) pieces so as to cover an 8×18 ft^2 room?

(b) Can you cut the 12×12 carpet into two (2) pieces so as to cover a 9×16 ft^2 room?

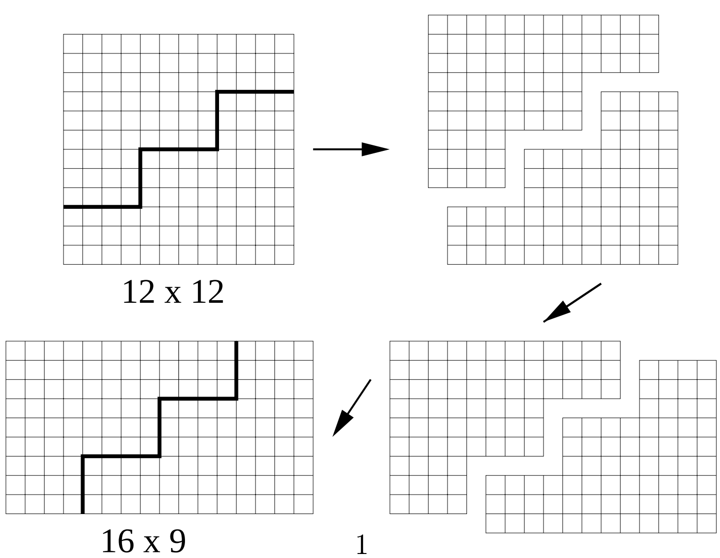
SOLUTION: A first attempt might be to try just one straight cut. However we soon discover that for such a cut to allow us to glue the two pieces together and get a rectangular carpet, we will have to be make it halfway. We get two pieces of dimensions 6×12 ft^2, which when glued together produce a carpet of length 6×24 ft^2, mmmmm, not the right dimensions.

(a) Next thing that comes to mind (at least to mine) is to try an L-shaped cut. This time we discover that the L-shaped cut that works must be symmetric, as pictured below. These two pieces can be reassembled together to produce the desired 8×18 carpet.



We realize this is the only possible cut that after reassembling covers an 8×18 room.

(b) Our L-shape is a 2-step staircase. How about a 3-step staircase? Again there is only one that can be reassembled into a rectangular shape, and it has the right dimensions 9×16!!



PROBLEM 2: Find real numbers A, B and C so that for all real numbers $x \neq 0, 3, -1$, the following identity holds

$$\frac{1}{x(x-3)(x+1)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+1}. \tag{1}$$

ANSWER: $A = -1/3$, $B = 1/12$ and $C = 1/4$.

SOLUTION: This is a standard problem on *partial fractions*. It is the basis of one of the integration techniques learned in a calculus course. For those of you that have already learned it, this will be an exercise. However for those of you who had never seen it before it requires some thinking.

Solution 1: There are several ways one can approach this problem. We could write the right hand side as a quotient of polynomials, the denominator being $x(x-3)(x+1)$, the same as in the left hand side, hence denominators can be cancelled out, and we are left with a polynomial identity:

$$1 = A(x-3)(x+1) + Bx(x+1) + Cx(x-3). \tag{2}$$

Now two polynomials are equal if and only if all their coefficients coincide. After simplifying and collecting terms on the right-hand-side in our case, we get,

$$0x^2 + 0x + 1 = (A+B+C)x^2 + (-2A+B-3C)x - 3A.$$

This translates into a system of 3 linear equations in the 3 unknowns A, B, and C, namely:

$$\begin{aligned} A+B+C &= 0, \\ -2A+B-3C &= 0, \\ -3A &= 1. \end{aligned}$$

We deduce immediately from the third equation that $A = -1/3$. We plug this value into the first two equations an we are left with a system of two linear equations in two unknowns, namely:

$$\begin{aligned} B+C &= 1/3, \\ B-3C &= -2/3. \end{aligned}$$

Which we can solve with our favorite method and we conclude that $B = 1/12$ and $C = 1/4$.

Solution 2: Another approach will be to assume there is a solution, and substitute into the identity (1) three values of $x \neq 0, 3, -1$ to obtain another system of 3 linear equations in the 3 unknowns, and then solve it. For example, substituting in $x = -2, 1, 2$ leads to the following system:

$$\begin{aligned} -A/2 - B/5 - C &= -1/10, \\ A - B/2 + C/2 &= -1/4, \\ A/2 - B + C/3 &= -1/6. \end{aligned}$$

Which, when solved, leads to the same solution as before: $A = -1/3$, $B = 1/12$ and $C = 1/4$.

Solution 3: This is a combination of the previous two. We would like to substitute in (1) the values $x = 0, 3, -1$. We cannot because the factors x, $x-3$ and $x+1$ appear in the denominator. Instead we can substitute $x = 0, 3, -1$ into (2) and conclude that

$$1 = -3A, \quad 1 = 12B, \quad 1 = 4C \quad \Rightarrow \quad A = -1/3, \quad B = 1/12, \quad C = 1/4.$$

PROBLEM 3: A spider is standing at the center of the bottom of a glass. The spider wants to reach a delicious ant that is standing on the rim of the glass. Assume the spider walks at constant speed and the ant, unaware of the danger, does not move.

(a) Suppose the glass is cylindrical of radius 1 unit and height 2 units. What distance should the spider walk to have her meal as quickly as possible?

(b) Suppose the glass now has a square base of side 2 units and height 2 units. The ant is standing in one of the top corners of the glass, and the spider is still at the center of the base. What distance should the spider walk to have her meal as quickly as possible?

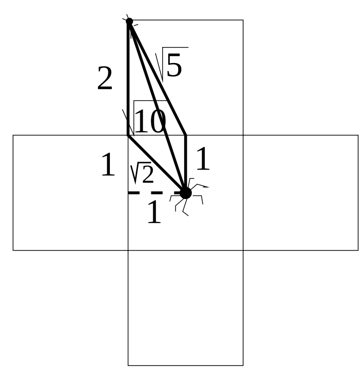
ANSWER: (a) The spider has to walk 3 units on the glass. (b) The spider has to walk $\sqrt{10}$ units on the glass.

SOLUTION: Since the spider walks at a constant speed, to get as fast as possible to its meal, means to find the shortest path. If the spider could shoot its web like spiderman to where the ant is, then the shortest path would be the straight line joining the center of the bottom of the glass and the point on the rim where the ant is. However the spider does not have superpowers and would have to walk on the glass, hence we are searching for the shortest path on the glass.

Had the spider had the spiderman powers, the shortest path on the cylindrical glass will have length $\sqrt{5}$ (the hypothenuse of a right triangle with side lengths 1 and 2). The shortest path along a web line on the square based glass will have length $\sqrt{6}$ (the hypothenuse of a right triangle with side lengths $\sqrt{2}$ and 2).

(a) The spider must reach the bottom of the cylindrical wall before it starts climbing the wall. Being at the center on the circular base, the shortest path to its boundary is to walk along a radius. Among all possible points on the boundary of the circular base, the one that is directly below the ant is the closest to the ant. The spider should walk 1 unit along a radius, then 2 units along the vertical path from the boundary of the circular base to the rim of the glass where the ant is. The total length of the shortest path will be 3 units.

(b) Here we might try the same approach as before. Walk along a straight line towards the corner of the glass where the ant is, then walk 2 units along the vertical line towards the ant. This path has a total length of $2 + \sqrt{2}$ units, because the straight line towards the corner is half the diagonal of square of sidelength 2. We could also take the shortest path to a point on the base of the wall nearest to the corner where the ant is, and then walk on a straight line on the wall of the glass towards the ant. This path has length $1 + \sqrt{5}$, which is smaller than $2 + \sqrt{2}$. This is not the shortest path either. We can see this by flattening the glass, and joining the center of the square base to a "corner" on the rim of the glass with a straight line. That is the shortest path. Its length is $\sqrt{10}$, and can be computed by the Pythagorean theorem since it is the hypothenuse of a right triangle with sides 1 and 3 units long.



PROBLEM 4: Observe that 4 can be expressed as the sum of non-zero natural numbers in 8 ways, taking into account the order of the terms:

$$4,\ \ 3+1,\ \ 1+3,\ \ 2+2,\ \ 2+1+1,\ \ 1+2+1,\ \ 1+1+2,\ \ 1+1+1+1.$$

(a) How many such expressions are there for 6?

(b) How many such expressions are there for 2004?

ANSWER: (a) We can write 6 as a sum of natural numbers in $32 = 2^5$ ways.
(b) We can write 2004 as a sum of natural numbers in 2^{2003} ways.

SOLUTION: (a) We can simply list all possibilities: 6, $5+1$, $1+5$, $4+2$, $2+4$, $3+3$, $4+1+1$, $1+4+1$, $1+1+4$, $3+2+1$, $3+1+2$, $2+3+1$, $2+1+3$, $1+2+3$, $1+3+2$, $2+2+2$, $3+1+1+1$, $1+3+1+1$, $1+1+3+1$, $1+1+1+3$, $2+2+1+1$, $2+1+2+1$, $2+1+1+2$, $1+2+2+1$, $1+2+1+2$, $1+1+2+2$, $2+1+1+1+1$, $1+2+1+1+1$, $1+1+2+1+1$, $1+1+1+2+1$, $1+1+1+1+2$, $1+1+1+1+1+1$. There are 32 such decompositions.

However, for part (b), listing all possibilities is out of question. A different method must be found.

(b) Let $S(n)$ be the number of decompositions of n into sums of natural numbers where the order matters. So far we know $S(4) = 8 = 2^3$, $S(6) = 32 = 2^5$. Had we also listed all possibilities for 1,2,3 and 5, we would have encountered that $S(1) = 1$, $S(2) = 2$, $S(3) = 4 = 2^2$, and $S(5) = 16 = 2^4$. We could then guess that $S(n) = 2^{n-1}$, hence $S(2004) = 2^{2003}$. This guess is correct, but requires a proof.

Solution 1 (Prof. L.-S. Hahn): This problem has a very short solution. Consider that n balls are arranged in line. Here are 6 balls ($\circ$)

$$\circ\quad\circ\quad\circ\quad\circ\quad\circ\quad\circ$$

And in each of the spaces between two balls, you may insert a divider. For example,

$$\circ\quad\circ\,|\,\circ\quad\circ\quad\circ\,|\,\circ$$

represents $2+3+1$. It should be clear that you can obtain any decomposition of 6 this way, and given any decomposition, you get one of these representations. Now with n balls, there are $n-1$ spaces and for each of these spaces, you have the option of whether to insert a divider or not. So all together, you have 2^{n-1} variations.

Solution 2: We will argue by induction. Suppose we know that $S(k) = 2^{k-1}$ for all $0 < k \leq n$. We would like to show that $S(n+1) = 2^n$. Clearly $S(1) = 1 = 2^0 = 2^{1-1}$ this initializes the induction. Let us count in how many ways we can write $n+1$ as a sum of non-zero natural numbers so that the last number is one: $(n)+1$ is the first one, and instead of listing all other decompositions with the same property, it suffices to note that to find them all we have to do is replace (n) by any of its 2^{n-1} decompositions given by the induction hypothesis. Similarly if we wanted to list all decompositions that end with the number two: $(n-1)+2$ is the first one, and then we will have all others by replacing $(n-1)$ by any of its 2^{n-2} decompositions. Keep going until we finish writing out all decompositions that end with the number n, there is only one of those $1+n$. Let us not forget the decomposition consisting of just the number itself, $(n+1)$. These are all possible decompositions of $n+1$, no more, no less. Here is the listing,

Decomposition ending in k	# of such decompositions
$(n)+1$	2^{n-1} ways
$(n-1)+2$	2^{n-2} ways
$(n-2)+3$	2^{n-3} ways
$\vdots$	$\vdots$
$(n+1-k)+k$	2^{n-k} ways
$\vdots$	$\vdots$
$3+(n-2)$	2^2 ways
$2+(n-1)$	2 ways
$1+n$	1 way
$(n+1)$	1 way

All we have to do is add up the right column to get the total number of decompositions for $n+1$:

$$S(n+1) = 1 + 1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-3} + 2^{n-2} + 2^{n-1} = 2^n.$$

To compute the above sum we can again argue by induction. Clearly $S(1) = 1 = 2^0$, $S(2) = 1 + 1 = 2$. Assume $S(n) = 2^{n-1}$, the objective is to show that $S(n+1) = 2^n$. But $S(n+1) = S(n) + 2^{n-1} = 2^{n-1} + 2^{n-1} = 2^n$, and we are done. The sum we want to compute is $S(2004) = 2^{2003}$.

Some of you might have recognized a *geometric sum* of the form

$$G(n) = 1 + r + r^2 + r^3 + \cdots + r^n,$$

with $r = 2$. To find a closed formula for such sums, you can use the above reasoning, or you can notice that if we multiply $G(n)$ by r,

$$rG(n) = r + r^2 + r^3 + \cdots + r^n + r^{n+1},$$

and subtract $G(n)$, then most terms will cancel out except for two of them, that is,

$$rG(n) - G(n) = (r-1)G(n) = r^{n+1} - 1 \quad \text{which implies} \quad G(n) = \frac{r^{n+1}-1}{r-1}, \quad r \neq 1.$$

In the case $r = 2$, this says that $G(n) = 2^{n+1} - 1$. We want to calculate $S(n+1) = 1 + G(n-1) = 2^n$.

PROBLEM 5: Rational numbers, are numbers of the form p/q, where p and q are integers, $q \neq 0$. For example $3/4$, and $2 = 2/1$ are rational numbers, however $\sqrt{5}$ is not. For any real number a, we define $x = a^{1/3}$ to be the unique real number x such that $x^3 = a$. For example $(-8)^{1/3} = -2$, because $(-2)^3 = -8$.

(a) Is $(2+\sqrt{5})^{1/3} + (2-\sqrt{5})^{1/3}$ a rational number? If YES, which one? If NO, why?

(b) Is $(2+\sqrt{5})^{1/3} - (2-\sqrt{5})^{1/3}$ a rational number? If YES, which one? If NO, why?

ANSWER: (a) YES, it is the rational number 1.

(b) NO, because it is the rational number found in part (a) minus the irrational quantity $2(2-\sqrt{5})^{1/3}$. One can compute the value and is exactly $\sqrt{5}$!.

SOLUTION: A few comments are in order before giving you the solution of this problem. Given two rational numbers (fractions), we can add (subtract) them, multiply them, divide them (provided the denominator is not zero), and we will obtain another rational number. However the same is not true for irrational numbers (numbers that are not rational). You can add (subtract) two irrational numbers and obtain a rational number, for example: $\sqrt{5}-\sqrt{5}=0$. You can multiply (divide) two irrational numbers and get a rational number, for example: $\sqrt{5}\sqrt{5}=5$, $\sqrt{5}/\sqrt{5}=1$. However if you add (subtract) or multiply (divide) a rational and an irrational number, you will always get an irrational number(otherwise you will conclude that the irrational number is rational which is a contradiction).

Famous examples of irrational numbers are π, e, $\sqrt{2}$. The first two examples are *transcendental* numbers, the last one is not, it is an *algebraic* number (that is the root of a polynomial with integer coefficients, e.g. $x^2-2=0$. Note that all rational numbers are algebraic!). The problem stated that $\sqrt{5}$ is not a rational number. Here is a proof of this fact: assume $\sqrt{5}$ is rational, $\sqrt{5}=p/q$, where p and q are positive integers. Squaring both sides and clearing denominators we get $5p^2=q^2$. Notice that the prime factor 5 appears an odd number of times on the left hand side, and an even number of times on the right hand side, this is a contradiction, since the prime decomposition of a number is unique. Thus, $\sqrt{5}$ cannot be a rational number. With this fact on hand it should be clear that $u=(2+\sqrt{5})^{1/3}$ is not a rational number, because if it was, we could solve for $\sqrt{5}=(u^3-2)$ and we will conclude that $\sqrt{5}$ is rational, which is a contradiction. Similarly, $v=(2-\sqrt{5})^{1/3}$ is not a rational number. Now adding or subtracting two irrational numbers, may or may not be rational. Our job is to discover what is the answer when adding (subtracting) u and v. Note that if

$$x=u+v, \qquad y=u-v,$$

then $x+y=2u=2(2+\sqrt{5})^{1/3}$ which is an irrational number. Therefore, x and y cannot be both rational numbers, because otherwise so will be their sum, and is not. Either x and y are both irrational, or one is rational and the other is not.

Solution 1 (Prof. L.-S. Hahn): (a) We are denoting by $x=(2+\sqrt{5})^{1/3}+(2-\sqrt{5})^{1/3}$ the first quantity we are trying to decide is rational or not. Since there are cube roots involved in its definition, let us raise the whole quantity to the third power and see what happens[1]:

$$\begin{aligned}
x^3 &= \left[(2+\sqrt{5})^{1/3}+(2-\sqrt{5})^{1/3}\right]^3 \\
&= 2+\sqrt{5}+3(2+\sqrt{5})^{2/3}(2-\sqrt{5})^{1/3}+3(2+\sqrt{5})^{1/3}(2-\sqrt{5})^{2/3}+2-\sqrt{5} \\
&= 4+3(2+\sqrt{5})^{1/3}(2-\sqrt{5})^{1/3}[(2+\sqrt{5})^{1/3}+(2-\sqrt{5})^{1/3}] \\
&= 4+3\left[(2+\sqrt{5})(2-\sqrt{5})\right]^{1/3}x \;=\; 4+3(4-5)^{1/3}x \\
&= 4+3(-1)^{1/3}x=4-3x.
\end{aligned}$$

From this calculation we conclude that the real number x must be a solution of the cubic equation

$$x^3+3x-4=0.$$

By inspection we conclude that $x=1$ is a solution of the above equation. Furthermore

$$x^3+3x-4=(x-1)(x^2+x+4),$$

so the other two solutions are roots of the quadratic equation $x^2+x+4=0$, which is quickly seen not to have any real roots. Therefore we conclude that $(2+\sqrt{5})^{1/3}+(2-\sqrt{5})^{1/3}$ must be the rational number 1.

[1]Remember that $(a+b)^3=a^3+3a^2b+3ab^2+b^3$, and $(a+b)(a-b)=a^2-b^2$.

(b) This automatically implies that

$$y = (2+\sqrt{5})^{1/3} - (2-\sqrt{5})^{1/3} \tag{3}$$

is not a rational number by the observation made right before the beginning of this solution. That is, if y is rational, then so is $x+y = 1+y$, but $x+y = 2(2+\sqrt{5})^{1/3}$ which we have already explained is an irrational number.

We could have proceeded like in part (a). Let us cube both sides of (3) and we obtain

$$\begin{aligned} y^3 &= 2\sqrt{5} - 3(2+\sqrt{5})^{1/3}(2-\sqrt{5})^{1/3}[(2+\sqrt{5})^{1/3} - (2-\sqrt{5})^{1/3}] \\ &= 2\sqrt{5} + 3y \end{aligned}$$

Therefore, $y^3 - 3y = 2\sqrt{5}$. If y is rational, then the left-hand side is rational, but the right-hand side is not rational, and we obtain a contradiction. We can stop here, since we have answered the question. However, we can continue as in part (a). The real number y must be a solution of the equation $y^3 - 3y - 2\sqrt{5} = 0$. By inspection we conclude that $y = \sqrt{5}$ is a solution[2]. Furthermore,

$$y^3 - 3y - 2\sqrt{5} = (y - \sqrt{5})(y^2 + \sqrt{5}y + 2),$$

and the other two solutions are roots of the quadratic equation $y^2 + \sqrt{5}y + 2 = 0$, which is quickly seen not to have any real roots. Therefore we conclude that $(2+\sqrt{5})^{1/3} - (2-\sqrt{5})^{1/3}$ must be the irrational number $\sqrt{5}$.

Solution 2 (Prof. Alex Stone): The real cube roots of $(2 \pm \sqrt{5})$ can be computed explicitly as follows. Assume that $(2+\sqrt{5})^{1/3} = a + b\sqrt{5}$, for a, b rational numbers. Then cube both sides and collect terms,

$$2 + \sqrt{5} = [a^3 + 15ab^2] + [3a^2b + 5b^3]\sqrt{5},$$

because the quantities in brackets on the right hand side are rationals, we must have,

$$\begin{aligned} a^3 + 15ab^2 &= 2, \\ 3a^2b + 5b^3 &= 1. \end{aligned}$$

We are lucky because by inspection we can find a real solution to this system, namely, $a = b = 1/2$. In which case, $(2+\sqrt{5})^{1/3} = \frac{1}{2}(1+\sqrt{5})$. Similarly, we find that $(2-\sqrt{5})^{1/3} = \frac{1}{2}(1-\sqrt{5})$. (Some of you recognized the cube roots by checking directly that $\left(\frac{1}{2}(1 \pm \sqrt{5})\right)^3 = 2 \pm \sqrt{5}$.) Substituting the real roots, we have

(a) $(2+\sqrt{5})^{1/3} + (2-\sqrt{5})^{1/3} = \dfrac{1}{2}(1+\sqrt{5}) + \dfrac{1}{2}(1-\sqrt{5}) = 1.$

(b) $(2+\sqrt{5})^{1/3} - (2-\sqrt{5})^{1/3} = \dfrac{1}{2}(1+\sqrt{5}) - \dfrac{1}{2}(1-\sqrt{5}) = \sqrt{5}.$

[2] A solution for the cubic in (a) was easy to guess. For the equation $y^3 - 3y - 2\sqrt{5} = 0$ it seems obvious once we are told what the solution is. But even if we had not been able to guess, there are methods to solve cubic equations, in particular *depressed cubics* where the y^2 term is missing. Prof. Alex Stone provided the following historical footnote: These equations were originally studied by the 16th century Italians (Tartaglia, Cardano, et al). Their method of solution was to make a substitution which transformed the cubic into a 6th degree equation, but quadratic in form. The substitution $y = z + \frac{1}{z}$ in this case leads to $z^6 - 2\sqrt{5}z^3 + 1 = 0$. Solving the quadratic equation in z^3, we obtain its solutions $z_1^3 = \sqrt{5} + 2$ and $z_2^3 = \sqrt{5} - 2$. Thus Tartaglia's problem would have been to find some cube roots. In the next solution we will find the real cube roots to be $z_1 = \frac{1}{2}(\sqrt{5}+1)$, $z_2 = \frac{1}{2}(\sqrt{5}-1)$, which happen to be reciprocals of each other ($z_2 = 1/z_1$), therefore a real solution to the cubic equation will be given by $\sqrt{5} = z_1 + \frac{1}{z_1} = z_1 + z_2$.

Solution 3 (11th grader Dimitar Bounov from United World Colleges): This we think is the best way to compute the value of the difference:

$$\begin{aligned}(2+\sqrt{5})^{1/3}-(2-\sqrt{5})^{1/3} &= \sqrt{\left((2+\sqrt{5})^{1/3}-(2-\sqrt{5})^{1/3}\right)^2}\\ &= \sqrt{(2+\sqrt{5})^{2/3}-2(2+\sqrt{5})^{1/3}(2-\sqrt{5})^{1/3}+(2-\sqrt{5})^{2/3}}\\ &= \sqrt{(2+\sqrt{5})^{2/3}+2(2+\sqrt{5})^{1/3}(2-\sqrt{5})^{1/3}+(2-\sqrt{5})^{2/3}-4(2+\sqrt{5})^{1/3}(2-\sqrt{5})^{1/3}}\\ &= \sqrt{\left((2+\sqrt{5})^{1/3}+(2-\sqrt{5})^{1/3}\right)^2-4[(2+\sqrt{5})(2-\sqrt{5})]^{1/3}}\\ &= \sqrt{1-4(4-5)^{1/3}} = \sqrt{1-4(-1)} = \sqrt{5}.\end{aligned}$$

Where in the last equalities we have used the result of part (a), and the fact that $(-1)^{1/3}=-1$.

Solution 4: This solution is based in the following observation

$$(\sqrt{5}+2)^{1/3}(\sqrt{5}-2)^{1/3}=\left[(\sqrt{5}+2)(\sqrt{5}-2)\right]^{1/3}=(5-4)^{1/3}=1.$$

That is $(\sqrt{5}-2)^{1/3}$ is the reciprocal of $(\sqrt{5}+2)^{1/3}$. Denote by $z=(2+\sqrt{5})^{1/3}$, then

$$\frac{1}{z}=(\sqrt{5}-2)^{1/3}=-(2-\sqrt{5})^{1/3}.$$

We are trying to decide whether $y=z+\frac{1}{z}$ is rational. It turns out that the quantity $y_n=z^n+\frac{1}{z^n}$ can be written as a polynomial of degree n in the variable $y=z+\frac{1}{z}$ with INTEGER COEFFICIENTS!![3] This implies that if y is rational so will be y_n for all n. But for $n=3$, $y_3=2\sqrt{5}$ which is NOT rational. Therefore y cannot be rational. For our purposes, all we need is to verify that $y_3=z^3+\frac{1}{z^3}$ can be written as a polynomial in $y=z+\frac{1}{z}$ with integer coefficients, in fact,

$$2\sqrt{5}=y_3=z^3+\frac{1}{z^3}=\left(z+\frac{1}{z}\right)^3-3\left(z+\frac{1}{z}\right)=y^3-3y.$$

Notice that this reduces to the argument used in Solution 1.

PROBLEM 6:

(a) A chord of length ℓ divides the interior of a circle of radius r into two regions. In each region, draw the largest possible circle so that the centers of the three circles are collinear. Express the area of the region that is inside the big circle (of radius r) but outside the two small circles in terms of r and ℓ.

(b) State and solve the 3-dimensional version of part (a). Remember that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

ANSWER: (a) The area of the region is $\frac{\pi\ell^2}{8}$ units^2.

(b) The volume of the region is $\frac{\pi r\ell^2}{4}$ units^3.

[3] Verify this statement. You could try an argument by induction!

SOLUTION 1: (a) Let r_1 and r_2 be the radii of the smaller circles. The area we wish to compute is the area of the larger circle minus the area of the smaller circles, that is,

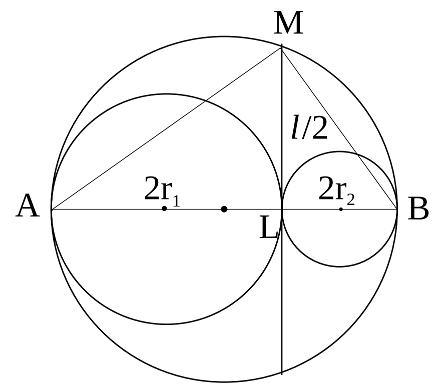
$$\text{Area} = \pi(r^2 - r_1^2 - r_2^2).$$

Notice that $r = r_1 + r_2$, and that

$$r^2 = r_1^2 + 2r_1r_2 + r_2^2.$$

Hence Area$= 2\pi r_1 r_2$. We will be done if we can get hold of the product r_1r_2.

Let L be the intersection point of the chord ℓ and the diameter of the big circle, and M one of the intersection points of the chord ℓ and the big circle. Let A and B be the endpoints of the diameter of the big circle. The triangle $\triangle ALM$ is similar to $\triangle BML$, therefore,



$$\frac{2r_1}{\ell/2} = \frac{\ell/2}{2r_2},$$

that is, $r_1r_2 = \dfrac{\ell^2}{16}$.

We conclude that Area $= \dfrac{\pi\ell^2}{8}$.

Notice that, if the length of the chord is ℓ, the area is the same, regardless of the radius r of the large circle. Which is quite surprising, since r can be arbitrarily large.

(b) Statement of the problem: *A plane intersects a sphere of radius r at a circle of diameter ℓ, and divides the sphere in two regions. In each region place the largest possible sphere so that the three centers are collinear. Express the volume of the region that is inside the big sphere but outside the smaller ones in terms of r and ℓ.*

This time the volume we want to compute is given by

$$\text{Volume} = \frac{4}{3}\pi(r^3 - r_1^3 - r_2^3),$$

where r_1 and r_2 are the radii of the smaller spheres. In this setting it is still true that $r = r_1 + r_2$. And the relation found in part (a) still holds in this case, by noting that if we cut with a plane that contains the three centers, then the section is the two dimensional picture in (a). That is, $r_1r_2 = \ell^2/16$. The following calculation holds:

$$r^3 = (r_1 + r_2)^3 = r_1^3 + 3r_1^2r_2 + 3r_1r_2^2 + r_2^3 = r_1^3 + 3r_1r_2(r_1 + r_2) + r_2^3.$$

That is,

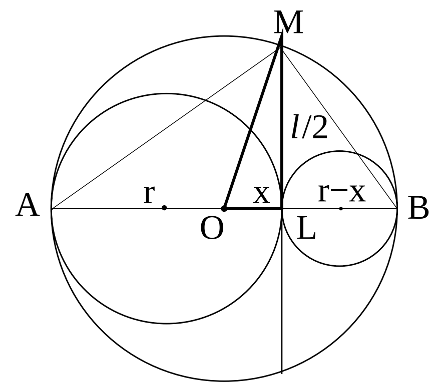
$$r^3 - r_1^3 - r_2^3 = 3\frac{\ell^2}{16}r.$$

Therefore the volume we are trying to find is given by

$$\text{Volume} = \frac{\pi r\ell^2}{4}.$$

SOLUTION 2 (Albuquerque Academy):

(a) Denote by O the center of the large circle, and L the intersection point of the chord ℓ and the diameter AB of the large circle. Let $x = OL$, then



$$2r_1 = r + x, \qquad 2r_2 = r - x.$$

We can express x in terms of r and ℓ, in fact, from the right triangle OLM we deduce that,

$$x = \sqrt{r^2 - \frac{\ell}{2}^2}.$$

Finally,

$$\begin{aligned} \text{Area} &= \pi(r^2 - r_1^2 - r_2^2) = \pi\left[r^2 - \left(\frac{r+x}{2}\right)^2 - \left(\frac{r-x}{2}\right)^2\right] \\ &= \pi\left[r^2 - \left(\frac{r + \sqrt{r^2 - \left(\frac{\ell}{2}\right)^2}}{2}\right)^2 - \left(\frac{r - \sqrt{r^2 - \left(\frac{\ell}{2}\right)^2}}{2}\right)^2\right]. \end{aligned}$$

The above formula can be simplified to get $\frac{\pi\ell^2}{8}$.

(b) The same formulae for r_1 and r_2, the radii of the smaller spheres hold, hence,

$$\begin{aligned} \text{Volume} &= \frac{4}{3}\pi(r^3 - r_1^3 - r_2^3) = \frac{4}{3}\pi\left[r^3 - \left(\frac{r+x}{2}\right)^3 - \left(\frac{r-x}{2}\right)^3\right] \\ &= \frac{4}{3}\pi\left[r^3 - \left(\frac{r + \sqrt{r^2 - \left(\frac{\ell}{2}\right)^2}}{2}\right)^3 - \left(\frac{r - \sqrt{r^2 - \left(\frac{\ell}{2}\right)^2}}{2}\right)^3\right]. \end{aligned}$$

Simplifying the above formula we get $\frac{\pi r\ell^2}{4}$.

PROBLEM 7: If θ is an angle in the first quadrant, and $3\cos\theta - 4\sin\theta = 2$, what is the value of $3\sin\theta + 4\cos\theta$?

ANSWER: $3\sin\theta + 4\cos\theta = \sqrt{21}$.

SOLUTION 1 (Prof. L.-S. Hahn): It is easy to see that

$$(3\sin\theta + 4\cos\theta)^2 + (3\cos\theta - 4\sin\theta)^2 = 25.$$

Therefore $(3\sin\theta + 4\cos\theta)^2 = 25 - 2^2 = 21$. It follows that $3\sin\theta + 4\cos\theta = \sqrt{21}$ because our angle is in the first quadrant.

SOLUTION 2 (Prof. L.-S. Hahn): Let

$$\begin{aligned} 3\sin\theta + 4\cos\theta &= x \\ -4\sin\theta + 3\cos\theta &= 2 \end{aligned}$$

Solving this system of simultaneous equations, we get

$$\sin\theta = \frac{3x-8}{25}, \quad \cos\theta = \frac{4x+6}{25}.$$

Because $\sin^2\theta + \cos^2\theta = 1$, we get

$$(3x-8)^2 + (4x+6)^2 = 25^2,$$

which simplifies to $x^2 = 21$. Therefore $x = \sqrt{21}$ because the angle is in the first quadrant.

SOLUTION 3: If none of the previous ideas occurred to you, you could still find the answer using the basic identity $\sin\theta = \sqrt{1-\cos^2\theta}$ (here we know $\sin\theta \geq 0$ because θ is in the first quadrant), and solving a quadratic equation in $\cos\theta$. More precisely, substituting in the given equation we get,

$$2 = 3\cos\theta - 4\sin\theta = 3\cos\theta - 4\sqrt{1-\cos^2\theta}.$$

Hence,

$$3\cos\theta - 2 = 4\sqrt{1-\cos^2\theta},$$

and squaring on both sides we get, after collecting terms, the following quadratic equation in $\cos\theta$,

$$25\cos^2\theta - 12\cos\theta - 12 = 0.$$

Solving the quadratic equation and choosing the only positive solution (since θ is in the first quadrant, $\cos\theta$ is positive) we get,

$$\cos\theta = \frac{2(3+2\sqrt{21})}{25}.$$

We can now evaluate $\sin\theta$,

$$\sin\theta = \sqrt{1-\cos^2\theta} = \frac{\sqrt{253-48\sqrt{21}}}{25}.$$

Finally we can substitute both values in the expression we were asked to compute to get,

$$3\sin\theta + 4\cos\theta = \frac{3\sqrt{253-48\sqrt{21}} + 8(3+2\sqrt{21})}{25}.$$

Exercise: Verify that this quantity is equal to $\sqrt{21}$.

If we had used the identity $\sin\theta = \sqrt{1-\cos^2\theta}$ instead, we would have solved the following quadratic equation in $\sin\theta$,

$$25\sin^2\theta + 16\sin\theta - 5 = 0$$

which implies $\sin\theta = \dfrac{-8+3\sqrt{21}}{25}$. From $\cos\theta = \sqrt{1-\sin^2\theta}$, we get $\cos\theta = \dfrac{\sqrt{372+48\sqrt{21}}}{25}$. Substituting these values we would get,

$$3\sin\theta + 4\cos\theta = \frac{3(-8+3\sqrt{21}) + 4\sqrt{372+48\sqrt{21}}}{25},$$

which also simplifies to $\sqrt{21}$.

SOLUTION 4: If you are familiar with trigonometric identities for the cosine and sine of the sum of two angles, namely

$$\begin{aligned}\cos(\alpha+\theta) &= \cos\alpha\cos\theta - \sin\alpha\sin\theta,\\ \sin(\alpha+\theta) &= \cos\alpha\sin\theta + \sin\alpha\cos\theta,\end{aligned}$$

you might find the clue for the solution of the problem. What if we declare $\cos\alpha = 3$ and $\sin\alpha = 4$? Then $\cos(\alpha+\theta) = 2$... but there is a problem, the sine and cosine functions are never larger, in absolute value, than 1, so we are not allowed to make the above choices. However, we know the fundamental Pythagorean identity linking sine and cosine of the same angle, namely $\sin^2\alpha + \cos^2\alpha = 1$. If instead we choose

$$\cos\alpha = 3/5, \qquad \sin\alpha = 4/5,$$

then we are in business, both are smaller than one in absolute value, and $(3/5)^2 + (4/5)^2 = 9/25 + 16/25 = 25/25 = 1$! We can use the first trigonometric identity to conclude that

$$\cos(\alpha+\theta) = \frac{3}{5}\cos\theta - \frac{4}{5}\sin\theta = \frac{3\cos\theta - 4\sin\theta}{5} = \frac{2}{5}.$$

Using now the second identity we conclude that $3\sin\theta + 4\cos\theta = 5\sin(\alpha+\theta)$. We are done as long as we can figure out the value of $\sin(\alpha+\theta)$, given that $\cos(\alpha+\theta) = 2/5$. It is time to use again the Pythagorean identity to conclude that

$$\sin(\alpha+\theta) = \pm\sqrt{1-\cos^2(\alpha+\theta)} = \pm\sqrt{1-\left(\frac{2}{5}\right)^2} = \pm\sqrt{\frac{25-4}{25}} = \pm\frac{\sqrt{21}}{5}.$$

Mmmmmmmm... we are almost done, except, how do we decide on the sign? Well, there is a piece of information in the problem that we have not used so far: *θ is an angle in the first quadrant*, and so is α. Hence $\alpha+\theta$ must be in the first or second quadrants, never in the third or fourth, thus we must choose the positive sign, and $3\sin\theta + 4\cos\theta = 5\sin(\alpha+\theta) = \sqrt{21}$.

For those of you familiar with complex numbers, you could solve the problem by noticing that the first expression is the real part of the product of the complex numbers $z = 3 + 4i$ and $w = \cos\theta + i\sin\theta$, and the second expression is its imaginary part.

PROBLEM 8: A billiard table is represented by rectangle $ABCD$. We assume the balls move on straight lines until they hit a side of the table. A ball *reflects* off a side according to the rule: *if ℓ is the line perpendicular to the side of the table on which the ball hits and passes through the hit point, then the angle between ℓ and the incident line is equal to the angle between ℓ and the reflection line.*

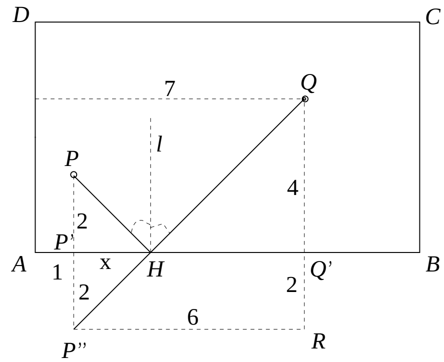
Let us assume we have balls sitting at points P and Q on the billiard table. Assume the table is 6×10 units^2, AB has length 10 units. To locate P, move 1 unit to the right of A, then 2 units up. To locate Q, move 7 units to the right of A, then 4 units up.

(a) You want to hit the ball at P against side AB so that it reflects and hits Q. Let H be the point on side AB you should aim to hit. What is the distance from A to H?

(b) You want the ball at P to first hit on side AB, then on side BC and then the ball at Q. If the bouncing points on sides AB and BC are respectively F and G, what are the distances from A to F and from B to G?

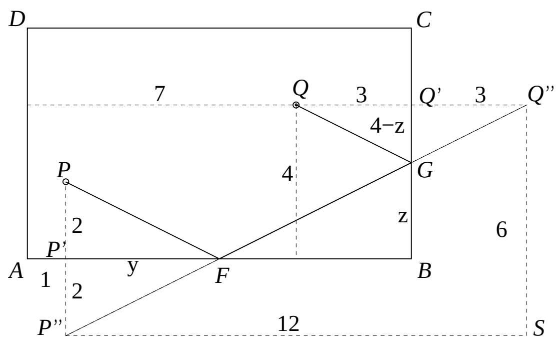
ANSWER: (a) $AH = 3$ units. **(b)** $AF = 5$ units, $BG = 5/2 = 2.5$ units.

SOLUTION 1: **(a)** Denote by P' the foot of the perpendicular from P to the side AB. Denote by P'' the point symmetric to P with respect to the side AB. Notice that $PP' = P'P'' = 2$. Denote by R the intersection point of the line parallel to side AB going through P'', and the line perpendicular to side AB going through Q. Notice that, $QR = 6 = P''R$. The triangles $P''RQ$ and $HP'P''$ are similar, therefore,



$$P'H = P'P'' = 2,$$

and $AH = P'H + 1 = 3$ units.



(b) Let P' and P'' be as in part (a). Denote by Q' the foot of the perpendicular from Q to side BC. Denote by Q'' the point symmetric to Q with respect to side BC. Notice that $QQ' = Q'Q'' = 3$. Denote by S the intersection point of the line parallel to side AB going through P'', and the line perpendicular to side AB going through Q''. Notice that $P''S = 12$ and $SQ'' = 6$. Then the triangle $P''SQ''$ is similar to the triangles $FP'P''$ and $Q''Q'G$, therefore,

$$\frac{P'F}{P'P''} = \frac{P''S}{SQ''} = 2, \text{ and } \frac{Q'Q''}{Q'G} = \frac{P''S}{SQ''} = 2.$$

Hence $P'F = 2P'P'' = 4$ units, and $Q'G = Q'Q''/2 = 3/2$. But $AF = P'F + 1$, and $BG = 4 - Q'G$, so we conclude that $AF = 5$ units, and $BG = 5/2$ units.

SOLUTION 2 (Albuquerque Academy): **(a)** With the same notation as in Solution 1 (a). Let Q' be the foot of the perpendicular from Q to side AB. Triangles $P'HP$ and $Q'HQ$ are similar, hence

$$\frac{P'H}{P'P} = \frac{Q'H}{Q'Q}.$$

But $P'H = x$, $Q'H = 6 - x$, $PP' = 2$, and $Q'Q = 4$, thus $x/2 = (6 - x)/4$, and solving for x we get $x = 2$. Finally $AH = 1 + x = 3$.

(b) With the same notation as in Solution 1 (b). Triangles $P'FP$, BFG, and $Q'QG$ are similar. Also notice that $P'F = y$, $FB = 9 - y$, $BG = z$, and $GQ' = 4 - z$, hence

$$\frac{9-y}{z} = \frac{y}{2} = \frac{3}{4-z}.$$

Solving for z in the first equality we get $z = \frac{18}{y} - 2$, and in the second we get $z = 4 - \frac{6}{y}$. The two right hand sides are equal: $\frac{18}{y} - 2 = 4 - \frac{6}{y}$. Solving for y we conclude that $y = 4$, hence $z = BG = 4 - \frac{6}{4} = 5/2$, and $AF = 1 + y = 5$.

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Dear students: If you have any suggestions about the Contest, or if you have different solutions to any of this year's problems, please send them to:

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Remember that you can find information about past contests at:
http://www.math.unm.edu/math_contest/contest.html

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