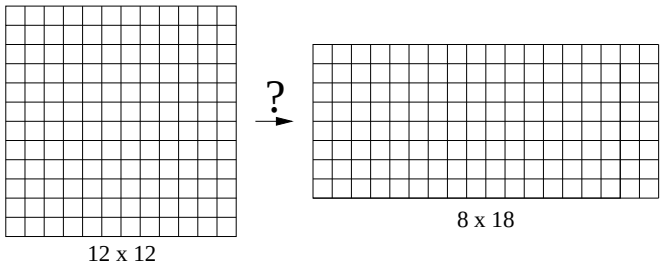
UNM-PNM STATEWIDE MATHEMATICS CONTEST XXXVII

November 13, 2004 FIRST ROUND THREE HOURS

PROBLEM 1: You are given a 12×12 ft^2 carpet.

(a) Can you cut the 12×12 carpet into two (2) pieces so as to cover an 8×18 ft^2 room?



(b) Can you cut the 12×12 carpet into two (2) pieces so as to cover a 9×16 ft^2 room?

PROBLEM 2: Find real numbers A, B and C so that for all real numbers $x \neq 0, 3, -1$, the following identity holds

$$\frac{1}{x(x-3)(x+1)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+1}.$$

PROBLEM 3: A spider is standing at the center of the bottom of a glass. The spider wants to reach a delicious ant that is standing on the rim of the glass. Assume the spider walks at constant speed and the ant, unaware of the danger, does not move.

(a) Suppose the glass is cylindrical of radius 1 unit and height 2 units. What distance should the spider walk to have her meal as quickly as possible?

(b) Suppose the glass now has a square base of side 2 units and height 2 units. The ant is standing in one of the top corners of the glass, and the spider is still at the center of the base. What distance should the spider walk to have her meal as quickly as possible?

PROBLEM 4: Observe that 4 can be expressed as the sum of natural numbers in 8 ways, taking into account the order of the terms:

$$4,\ \ 3+1,\ \ 1+3,\ \ 2+2,\ \ 2+1+1,\ \ 1+2+1,\ \ 1+1+2,\ \ 1+1+1+1.$$

(a) How many such expressions are there for 6?

(b) How many such expressions are there for 2004?

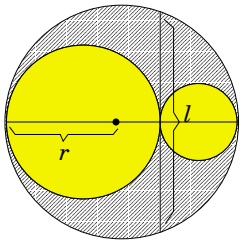
PROBLEM 5: Rational numbers, are numbers of the form p/q, where p and q are integers, and $q \neq 0$. For example $3/4$, and $2 = 2/1$ are rational numbers, however $\sqrt{5}$ is not. For any real number a, we define $x = a^{1/3}$ to be the unique real number x such that $x^3 = a$. For example $(-8)^{1/3} = -2$, because $(-2)^3 = -8$.

(a) Is $(2+\sqrt{5})^{1/3} + (2-\sqrt{5})^{1/3}$ a rational number? If YES, which one? If NO, why?

(b) Is $(2+\sqrt{5})^{1/3} - (2-\sqrt{5})^{1/3}$ a rational number? If YES, which one? If NO, why?

PROBLEM 6:

(a) A chord of length ℓ divides the interior of a circle of radius r into two regions. In each region, draw the largest possible circle so that the centers of the three circles are collinear. Express the area of the region that is inside the big circle (of radius r) but outside the two small circles in terms of r and ℓ.



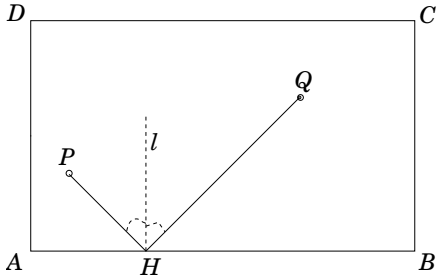
(b) State and solve the 3-dimensional version of part (a). Remember that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

PROBLEM 7: If θ is an angle in the first quadrant, and $3\cos\theta - 4\sin\theta = 2$, what is the value of $3\sin\theta + 4\cos\theta$?

PROBLEM 8: A billiard table is represented by rectangle $ABCD$. We assume the balls move on straight lines until they hit a side of the table. A ball *reflects* off a side according to the rule: *if ℓ is the line perpendicular to the side of the table on which the ball hits and passes through the hit point, then the angle between ℓ and the incident line is equal to the angle between ℓ and the reflection line.*

Let us assume we have balls sitting at points P and Q on the billiard table. Assume the table is 6×10 units^2, AB has length 10 units. To locate P, move 1 unit to the right of A, then 2 units up. To locate Q, move 7 units to the right of A, then 4 units up.

(a) You want to hit the ball at P against side AB so that it reflects and hits the ball at Q. Let H be the point on side AB you should aim to hit. What is the distance from A to H?



(b) You want the ball at P to first hit on side AB, then on side BC and then the ball at Q. If the bouncing points on sides AB and BC are respectively F and G, what are the distances from A to F and from B to G?