SOLUTIONS FOR UNM–PNM STATEWIDE MATHEMATICS CONTEST XXXVI

1. A fair coin is tossed 10 times. What is the probability that exactly five of the tosses come up heads and five of them tails?

The total number of possible outcomes is 2^{10} since each of the tosses must be either heads or tails. The total number of ways in which five of them can be heads is the combinatorial symbol

$$\binom{10}{5}.$$

We have

$$\binom{10}{5} = \frac{10!}{5!5!}.$$

Thus the answer is

$$\frac{10!}{5!5!2^{10}} = \frac{252}{1024} = \frac{63}{256}.$$

2. An integer–valued point in the xy–plane is a point (a, b) where both a and b are integers. How many integer–valued points are on or inside a circle of radius 4 centered at the origin?

So here we are counting the number of poins (x, y) where x and y are integers satisfying

$$\sqrt{x^2 + y^2} \leq 4.$$

One can run through the various possibilities: if $y = 0$ then x can be any number from -4 to 4 so this is 9 points. If $y = \pm 1$ then there are 7 possible values of x, ranging from -3 through $+3$. When $y = \pm 2$ there are still 7 possible values for x, from -3 through $+3$. For $y = \pm 3$ there are now 5 values for x, ranging from -2 through $+2$. And finally when $y = \pm 4$ the only possible value of x is 0. Adding all of this up gives

$$9 + 7 + 7 + 7 + 7 + 5 + 5 + 1 + 1 = 49.$$

3. Recall that $n!$ is the product of the first n positive integers, that is $2! = 2 \cdot 1$, $3! = 3 \cdot 2 \cdot 1$, and so on.

a. How many zeroes are at the end of 17!?

b. What is the smallest n such that $n!$ ends in exactly 37 zeroes?

Each zero at the end of $n!$ represents one time that 10 divides $n!$. So how many times will 10 divide 17!? Since $10 = 5 \cdot 2$ we will need to count how many times 5 divides 17! and then how many times 2 divides 17!. Looking at the number of factors in 17! that are divisible by

5 we find 3, namely 5,10, and 15. There are certainly more than 3 factors of 2 in 17! as the factor 16 alone has 4 two's. So there will be exactly 3 zeroes at the end of 17!.

To find the smallest n so that $n!$ ends in exactly 37 zeroes we'll look for the smallest n so that $n!$ is divisible by 5 exactly 37 times. Only every fifth positive integer is divisible by 5 so our answer must be divisible by 5. The number of 5's in $5k!$ grows linearly for small k but once $k = 5$ we have TWO factors of 5 in 25 and so 25! is divisible by 5^6. Similarly we find that 5^{12} divides 50!, 5^{18} divides 75!, 5^{24} divides 100!. When we reach 125! there will be 31 factors of 5, rather than just 30, because 125 has three factors of 5 rather than just two. Finally 150! is divisible by 5 exactly 37 times and is the smallest positive integer with this property. There are certainly more than 37 factors of 2 in 150! (in particular there are at least 75 since every other number is even) so the answer is 150.

4. a. Is 2003 a prime number?

b. What is the last digit of 2003^{2003}?

One checks that 2003 is a prime number by dividing by all primes less than or equal to 43 (the next smallest prime, 47, is larger than $\sqrt{2003}$ so if 2003 factors it must have a factor smaller than 47).

To find the last digit of 2003^{2003} note first that the last digit is unaffected by the 2000. Indeed, writing $2003 = 2000 + 3$ we have

$$2003^{2003} = (2000 + 3)^{2003}.$$

But in the expansion of $(2000+3)^{2003}$ all terms end in zero except the very last one, namely 3^{2003}. Thus we only need to find the last digit of 3^{2003}. If we compute the first 4 powers of 3 we find that they end in $3, 9, 7$, and 1 respectively. This same pattern will then continue: 3^5 ends in 3, 3^6 ends in 9, 3^7 ends in 7, and 3^8 ends in 1. More generally, if n is divisible by 4 then 3^n ends in 1. If n leaves a remainder of 1 when divided by 4 then 3^n ends in 3. If n leaves a remainder of 2 when divided by 4 then 3^n ends in 9. Finally if n leaves a remainder of 3 when divided by 4 then 3^n ends in 7. Since 2003 leaves a remainder of 3 when divided by 4, the answer to part b is 7.

5. Suppose $f(X) = aX^2 + bX + c$ is a quadratic polynomial and a, b, c are rational numbers. Suppose that $f(n)$ is an integer whenever n is an integer. Are a, b, and c necessarily integers? If not, give an example.

The answer here is no. The simplest polynomial with non–integer coefficients which always takes integer values is

$$f(X) = \frac{X^2}{2} + \frac{X}{2}.$$

The important point is that $X^2 + X$ is always even regardless of whether X is even or odd. One can see this clearly by noting that

$$X^2 + X = X(X + 1)$$

and either X or $X + 1$ must be even, making the product $X(X + 1)$ even.

6. John's father asks him to rake leaves and, in order to refine John's math skills, he offers John two possible choices of payment for his work:

A. one cent for the first bag, two cents for the second bag, four cents for the third bag, and so on: in other words 2^{n-1} cents for the n^{th} bag of leaves.

B. one dollar for the first bag, 2 dollars for the second bag, 3 dollars for the third bag, and so on: in other words n dollars for the n^{th} bag of leaves.

How many bags must John rake before option **A** becomes more profitable than option **B**?

Here one needs a couple of formulas, or else some patience working through options A and B one by one. So with option A after raking n bags John will be paid, in dollars,

$$.01(1 + 2 + \ldots + 2^{n-1}) = .01(2^n - 1).$$

On the other hand with option B raking n bags will give him, in dollars,

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

So we want to find the smallest n with

$$.01(2^n - 1) > \frac{n(n+1)}{2}.$$

Here one must use a little trial and error. From Problem 1 we have that $2^{10} = 1024$ so after 10 bags option A yields \$10.23. Meanwhile option B gives \$55 so the two quantities are starting to become close. When $n = 14$ option A gives a little over \$160 while option B gives \$105 and this is the smallest value of n for which A gives more than B.

7. a. What is the smallest number of coins (pennies, nickels, dimes, quarters, and half dollars) with which you can pay out any amount from 1 cent to 99 cents?

b. Suppose you can introduce coins of *any* denomination you wish. What is the smallest number of coins necessary to be able to pay out any amount from one cent to 99 cents?

For part a, the easiest way to see the anwer is to start with small amounts and work our way up. So the only way to pay out 4 cents is with 4 pennies so we need AT LEAST 4 pennies. To have more than 4 pennies, however, would be wasteful as we can start to use nickels after this. With 4 pennies and one nickel we can make very amount less than a dime. One we reach 10 cents there is no need to use more nickels and pennies so we start to use dimes. With 4 pennies, 1 nickel, and 1 dime we get every amount up to 19 cents. With 4 pennies, 1 nickel and 2 dimes we get up to 29 cents. At this point we introduce 1 quarter and then finally 1 half dollar to get every amount up to and including 99 cents. So we need a total of 9 coins. Note that instead of two dimes one could use 2 nickels instead.

For part b, we start with 1 penny which is necessary as we may need to pay out one cent. For 2 cents we could introduce another penny but it is better to introduce a two cent piece (did you know that the United States government made 2 cent pieces from 1864–1873?). This gives us every amount up through 3 cents and then we introduce a 4 cent piece (these never have existed but 3 cent pieces did for almost half of the 19th century). Next we introduce an 8 cent piece, a 16 cent piece, a 32 cent piece, and a 64 cent piece. To make change with this coinage would be equivalent to taking a number between 1 and 99 and writing it in base 2.

8. Farmer Brown has eight logs, each of length 10 feet. What is the *maximum* area which he can enclose with the logs? For example, he could make a rectangle of height 10 feet and width 30 feet or a square with each side having length 20 feet. In the first case, he has enclosed 300 square feet and in the second case 400 square feet. The second choice is of course better than the first but what is the *largest* area which Farmer Brown can enclose?

The maximum enclosed area will come from a regular octagon. To find its area note that it consists of one central square of dimensions 10 by 10, 4 rectangular regions of dimensions 10 by $\sqrt{50}$, and 4 triangular regions of with height and width equal to $\sqrt{50}$. This adds up to

$$100 + 40\sqrt{50} + 100 = 200 + 200\sqrt{2}.$$

Other acceptable answers to this question (and I leave this as a fun exercise for you to check that they are correct) are: $200 \tan(67.5)$ and $400\sqrt{2}\sin^2(67.5)$. A far more challenging exercise would be to PROVE that the regular octogon is, amonst all octogons with fixed perimeter, the one bounding the most area.