UNM–PNM STATEWIDE MATHEMATICS CONTEST XXXVI

February 14, 2004 SECOND ROUND THREE HOURS

1. We'll begin with the required problem about the current year 2004.

a. Give the prime factorization of 2004.

b. How many positive integers divide 2004 evenly?

2. Here is another problem concerning prime numbers.

a. Let $L(X) = 12X + 115$. Find the *smallest* integer $n \geq 1$ so that $L(n)$ is *not* a prime number.

b. Does there exist an integer $a \geq 1$ so that $an + 1$ is a prime number for all $n \geq 1$? Justify your answer.

c. Suppose $P(X) = a_d X^d + a_{d-1} X^{d-1} + \ldots + a_1 X + a_0$ is a polynomial where the coefficients $a_0, \ldots, a_d$ are integers, $a_d \neq 0$, and $d \geq 1$. Can $P(n)$ be a prime number for *all* integers $n \geq 1$ when the degree d is arbitrary? Justify your answer.

3. The sequence of Fibonacci numbers is:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

The first two elements of the sequence are 1 and then each successive member is obtained by adding the two previous elements: $F_n = F_{n-1} + F_{n-2}$ where F_n denotes the n^{th} number in the sequence.

a. Find F_{17}, the 17^{th} Fibonacci number.

b. Show that

$$\frac{F_2}{F_1} < \frac{F_4}{F_3} < \frac{F_6}{F_5} < \ldots,$$

i.e. show that for any $n > 0$ we always have $F_{2n}/F_{2n-1} < F_{2n+2}/F_{2n+1}$.

c. What value must

$$\frac{F_{n+1}}{F_n}$$

approach as n grows, *assuming that it does approach some value*?

4. David tosses a quarter in the air and watches it land on a checker board. Suppose that the length and width of each of the 64 squares in the checker board is exactly *twice* the diameter of the quarter and also suppose that the quarter lands *entirely* within the checker board, with equal likelihood at any point.

a. What is the likelihood that the quarter lands entirely within one of the 64 squares?

b. What is the likelihood that the quarter touches exactly three of the 64 squares?

5. Way back in 1901, Jacqueline's great grandfather deposited a brand new 1901 quarter from the San Francisco mint in the bank. This 1901 quarter is, however, a VERY rare coin, worth $40,000 today in 2004.

a. Assuming that the quarter earned 20% annual interest, would it be worth more or less than one dollar after 8 years?

b. Assuming that the quarter earned 10% annual interest for the entire 103 year period, would it be worth more or less than $40,000?

c. Assuming that the quarter earned 13% annual interest for the entire 103 year period, would it be worth more or less than $40,000?

Your answers in **a**, **b** and **c** must be justified.

6. Your school teacher presents you with the following problem: she gives you a hat, containing 5 slips of paper, each with a different number on it. You know *nothing* about the numbers: they could be of any size, positive or negative. You are asked to draw numbers successively out of the hat and look at them. The problem is to stop at the moment you have selected the *largest* of the 5 numbers. Of course, since you do not know what any of the numbers are in advance, it is impossible to solve this problem with *certainty.*

a. Suppose you employ the following method: you look at and discard one number which we will call A. Next you continue to draw until you find a number *larger* than A and stop here. What is the probability that you have stopped at the largest of the five numbers? Note that the definition of probability here is the total number of cases in which you are successful divided by the total number of all possible cases.

b. Assuming now that there are 100 slips of papers in the hat, each with a different number on it, give a method which allows you to stop at the moment you have selected the largest number more than 1/4 of the time. Of course you need to prove that the method will be successful more than 1/4 of the time.

7. An integer–valued point in the xy–plane is a point (a, b) where both a and b are integers. Let A_n denote the number of integer–valued points on or inside a circle of radius n centered at the origin. As n grows larger and larger what value will

$$\frac{A_n}{n^2}$$

approach? Justify your answer.